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Plate Estimates Involving Two Strain Parameters

Chester B. Sensenig

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Chester B. Sensenig

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Abstract

In this report estimates for the stresses and strains and their derivatives are derived for a thin plate. An isotropic material having a strain energy density function which is an analytic function of the strain invariants is considered, and the estimates are derived from the exact three dimensional theory of elasticity. These estimates are then used to derive approximate interior plate equations with error estimates.

The estimates obtained are of the same type as those derived by F. John [1] for shells, but the work here differs from that in [1] in that consideration is restricted to plates and two strain parameters are used rather than one.

The strain parameters ε_1 and ε_2 are chosen so that when the deformed plate is nearly flat ε_1 is small relative to ε_2 and for relatively large bending ε_2 is small relative to ε_1. Consequently, various simplifications in the plate equations are possible for either small bending or relatively large bending. Also certain vacuities in estimates are avoided by using the two strain parameters.

Introduction

In this report we derive estimates for the stresses and strains and their derivatives in a thin plate. Let u_1, u_2, and u_3 be the displacements parallel to the coordinate axes (Cartesian) of the particle which was at (x_1,x_2,x_3) in the undeformed plate, and let

$$e_{ij}(x_1,x_2,x_3) = \frac{1}{2}(u_{i,j}+u_{j,i}+u_{k,i}u_{k,j}) \tag{0.1}$$

be the strains where $_{,i}$ denotes $\partial/\partial x_i$, Latin indices assume the values 1,2,3 and repeated indices are summed over their range.

We consider a plate which in the undeformed state is bounded by the planes $x_3 = \pm h$ and by a verticle surface at the edge.

We define quantities $\sigma_{\alpha\beta}$ and $b_{\alpha\beta}$ in terms of the longitudinal strains on the middle surface by

$$\begin{aligned} \sigma_{\alpha\beta}(x_1,x_2) &= e_{\alpha\beta}(x_1,x_2,0) \\ b_{\alpha\beta}(x_1,x_2) &= e_{\alpha\beta,3}(x_1,x_2,0) \end{aligned} \tag{0.2}$$

where Greek indices take on the values 1,2.

Formulas with error estimates are obtained expressing the stresses and strains in the three dimensional plate in terms of the quantities $\sigma_{\alpha\beta}$ and $b_{\alpha\beta}$, and approximate differential equations with error estimates are obtained for the quantities $\sigma_{\alpha\beta}$ and $b_{\alpha\beta}$.

The theory used is the exact three dimensional theory for an arbitrary homogeneous isotropic material having a strain energy density function which is analytic in the strain invariants.

The estimates obtained are of the same type as those derived by F. John [1] for shells, but the work here differs from that in [1] in that two strain parameters are used in the estimates rather than one, separate estimates are obtained for the odd and even parts (considered as a function of x_3) of each function estimated, and consideration is restricted to plates.

Significant features of the results presented here are that the estimates give useful information in some cases where the corresponding estimates given in [1] are vacuous, and simplifications result when either one of the strain parameters is sufficiently small relative to the other.

We introduce non-dimensional stresses t_{ij} by letting

$$t_{ij} = \frac{1}{E} \frac{\partial W}{\partial e_{ij}} \tag{0.3}$$

where W is the strain energy density function (energy per unit undeformed volume), and E is Young's modulus for the material. If $\bar{S}$ is a surface in the deformed plate and S is the surface in the undeformed plate consisting of the same particles, then $Et_{ij}n_j\vec{g}_i$ is the stress vector (force per unit undeformed area) acting across $\bar{S}$ where (n_1,n_2,n_3) is the unit normal vector to S pointing to the particles exerting the force across $\bar{S}$, and $\vec{g}_i = \frac{\partial}{\partial x_i}(x_1+u_1,x_2+u_2,x_3+u_3)$ is a tangent vector to the curve consisting of the particles which were on a line parallel to the ith coordinate axis in the undeformed plate.

We restrict ourselves to deformations in which the stress vector is zero at the faces and the body force vector is zero throughout the plate. This gives

(0.4) $$t_{i3} = 0 \quad \text{for} \quad x_3 = \pm h \qquad (i = 1,2,3)$$

and the equilibrium equations [1,2]

(0.5) $$t_{ij,j} + 2e_{ik}t_{kj,j} + 2e_{ik,j}t_{kj} = \frac{1}{E} W_{,i} \qquad (i = 1,2,3).$$

Each of the functions t_{ij} can be expressed in a unique way as the sum of an even function of x_3 and an odd functions of x_3. We define $t_{ij\omega}$ $(\omega=1,2)$ to be the even part of t_{ij} if ω plus the number of threes among i,j is even. Otherwise $t_{ij\omega}$ is defined to be the odd part of t_{ij}, i.e. $t_{\alpha\beta 2}$, $t_{\alpha 31}$, and t_{332} are even functions of x_3 and $t_{\alpha\beta 1}$, $t_{\alpha 32}$, and t_{331} are odd functions of x_3. Similarly we define $e_{ij\omega}$.

Next we introduce the parameters in terms of which estimates are given. If $(\hat{x}_1,\hat{x}_2,\hat{x}_3)$ is any point in the undeformed plate, let D be the horizontal distance from $(\hat{x}_1,\hat{x}_2,\hat{x}_3)$ to the edge of the undeformed plate. We show that there is a fixed number $\theta_0 \leq 1$ depending only on the strain energy density function such that the estimates presented in the following are valid at $(\hat{x}_1,\hat{x}_2,\hat{x}_3)$ if

(0.6) $$\frac{h}{D} \leq \theta_0 \quad \text{and} \quad |e_{ij\omega}| \leq \theta_0^2$$

hold in the cylinder $(x_1-\hat{x}_1)^2 + (x_2-\hat{x}_2)^2 \leq D^2$, $|x_3| \leq h$ in the undeformed plate. We let C denote this cylinder. Let

$$\varepsilon_1 = \max |e_{ij1}|$$

(0.7) $$\varepsilon_2 = \begin{cases} \max\left(\frac{D^2}{h^2}\varepsilon_1^2, |e_{ij2}|\right) & \text{if } \varepsilon_1 \leq \frac{h^2}{D^2} \text{ and } \max|e_{ij2}| \leq \frac{h^2}{D^2} \\ \max(\varepsilon_1, |e_{ij2}|) & \text{otherwise;} \end{cases}$$

where the maximum is taken with respect to all values of the indices and all points in the cylinder C.

Let

(0.8) $$\theta = \max\left(\frac{h}{D}, \sqrt{\varepsilon_2}\right)$$
$$d = \frac{\theta_0}{\theta} h$$

The estimates will be given in terms of the parameters ε_1, ε_2, h, d, and θ.

In all cases $\max|e_{ij2}| \leq \varepsilon_2$. We show that this is an equality if $\varepsilon_1 \leq \max|e_{ij2}|$. The result follows immediately from the definition of ε_2 if $\varepsilon_1 > h^2/D^2$ or $\max|e_{ij2}| > h^2/D^2$. If $\varepsilon_1 \leq h^2/D^2$ and $\max|e_{ij2}| \leq h^2/D^2$, then $D^2/h^2\ \varepsilon_1^2 \leq \varepsilon_1 \leq \max|e_{ij2}|$ so that $\varepsilon_2 = \max|e_{ij2}|$. Thus in any event $\varepsilon_1 \leq \max|e_{ij2}|$ implies $\varepsilon_2 = \max|e_{ij2}|$ and $\varepsilon_1 \leq \varepsilon_2$. In fact we can have $\varepsilon_1 = 0$ when $\varepsilon_2 > 0$. This will occur if the horizontal displacements are even functions of x_3 and the vertical displacement is an odd function of x_3. In this case there is no bending of the plate.

We can also have $\varepsilon_1 \geq \varepsilon_2$. In particular, if $\varepsilon_1 \leq h^2/D^2$ and $\max|e_{ij2}| \leq D^2/h^2\ \varepsilon_1^2$, then $\varepsilon_2 = D^2/h^2\ \varepsilon_1^2 \leq \varepsilon_1$. In fact if $\varepsilon_1 << h^2/D^2$, then $\varepsilon_2 << \varepsilon_1$. However, from (0.7) it is clear that we can not have $\varepsilon_2 = 0$ and $\varepsilon_1 > 0$. We can have $\varepsilon_2 << \varepsilon_1$ if the horizontal displacements are nearly odd functions of x_3 and the vertical displacement is nearly an even function of x_3. This would correspond to relatively large bending of the plate.

The following are easily verified

$$
\begin{aligned}
&\theta \leq \theta_0 \leq 1\\
&\frac{h}{d} = \frac{\theta}{\theta_0} \leq 1\\
(0.9)\qquad &\varepsilon_2 \leq \theta^2 = \theta_0^2\,\frac{h^2}{d^2} \leq \frac{h^2}{d^2}\\
&\varepsilon_1 \leq \theta\sqrt{\varepsilon_2} \leq \theta^2\\
&|e_{ij1}| \leq \varepsilon_1\\
&|e_{ij2}| \leq \varepsilon_2
\end{aligned}
$$

where the last two lines hold in C.
$\theta \leq \theta_0$ follows from (0.6), (0.7).

If $\varepsilon_1 \le \frac{h^2}{D^2}$ and $\max|e_{ij2}| \le \frac{h^2}{D^2}$, then $\varepsilon_2 \ge \frac{D^2}{h^2}\varepsilon_1^2$ so that $\varepsilon_1 \le \frac{h}{D}\sqrt{\varepsilon_2} \le \theta\sqrt{\varepsilon_2}$. Otherwise $\varepsilon_1 \le \varepsilon_2 \le \theta\sqrt{\varepsilon_2}$ since $\varepsilon_2 \le \theta^2$. This completes the proof that $\varepsilon_1 \le \theta\sqrt{\varepsilon_2}$, and the remainder of (0.9) is immediate.

The first estimates which are obtained for the stresses and their derivatives are listed next in (0.10). An expression of the type $O(A)$ represents a quantity whose absolute value is less than or equal $\kappa|A|$ where κ is a positive constant whose value depends only on the strain energy density function. In (0.10) the numbers k_1, k_2, k_3 are any non-negative integers which make the notation meaningful, and there is not necessarily any relationship between their values on one line and their values on the next. Thus $k_1+k_2 = k$ on some lines and $k_1+k_2+k_3 = k$ on other lines. The first line in (0.10) gives an estimate for all derivatives of all stresses, and the remaining lines give sharper estimates for some derivatives of some stresses. We should also state that k may take on the value zero on each line of (0.10). A statement such as $(i,j) = (\alpha,\beta)$ means that i and j take on the values 1,2.

$$
\frac{\partial^k t_{ij\omega}}{\partial x_1^{k_1}\partial x_2^{k_2}\partial x_3^{k_3}} = \begin{cases} \mathcal{O}\left(\frac{\varepsilon_\omega}{d^k}\right) & \begin{cases} \text{if } k_3 = 0 \\ \text{if } k_3=1 \text{ and } i=3 \text{ or } j=3 \\ \text{if } k_3 \text{ is even and } (i,j)= (\alpha,\beta) \text{or} (3,3) \\ \text{if } k_3 \text{ is odd and } (i,j)=(\alpha,3) \end{cases} \\ \mathcal{O}\left(\frac{\varepsilon_\omega}{h\, d^{k-1}}\right) & \text{otherwise} \end{cases}
$$

$$
\begin{aligned}
& \frac{\partial^k t_{\alpha 31}}{\partial x_1^{k_1}\partial x_2^{k_2}} = \mathcal{O}\left(\frac{h}{d^{k+1}}\,\varepsilon_1\right) \\
(0.10)\quad & \frac{\partial^k t_{\alpha 32}}{\partial x_1^{k_1}\partial x_2^{k_2}\partial x_3^{k_3}} = \mathcal{O}\left(\frac{h^{3-k_3}}{d^{k+3-k_3}}\,\varepsilon_2\right) \qquad \text{for } k_3 = 0,1,2 \\
& \frac{\partial^k t_{33\omega}}{\partial x_1^{k_1}\partial x_2^{k_2}\partial x_3^{k_3}} = \mathcal{O}\left(\frac{h^{2-k_3}}{d^{k+2-k_3}}\,\varepsilon_\omega\right) \qquad \text{for } k_3 = 0,1 \\
& \frac{\partial^k t_{ij2}}{\partial x_1^{k_1}\partial x_2^{k_2}\partial x_3^{k_3}} = \mathcal{O}\left(\frac{h}{d^{k+1}}\,\varepsilon_2\right) \quad \begin{cases} \text{if } k_3 \text{ is odd and} \\ \quad (i,j)=(\alpha,\beta) \text{ or } (3,3) \\ \text{if } k_3 \text{ is even and} \\ \quad (i,j) = (\alpha,3)\ . \end{cases}
\end{aligned}
$$

Each of the inequalities in (0.10) is valid at the particle which was at $(\hat{x}_1,\hat{x}_2,\hat{x}_3)$ in the undeformed plate. Actually, each inequality in (0.10) is valid at all particles which occupied a cylinder in the undeformed plate which is concentric with C and whose radius depends only on the strain energy density function and on the value of k in the inequality.

Similar estimates are obtained for the derivatives of the strains. Namely,

$$\frac{\partial^k e_{ij\omega}}{\partial x_1^{k_1}\partial x_2^{k_2}\partial x_3^{k_3}} = \begin{cases} \mathcal{O}\left(\dfrac{\varepsilon_\omega}{d^k}\right) & \begin{array}{l}\text{if } k_3 = 0 \\ \text{if } k_3 \text{ is even and } (i,j)=(\alpha,\beta)\text{or}(3,3) \\ \text{if } k_3 \text{ is odd and } (i,j)=(\alpha,3)\end{array} \\ \mathcal{O}\left(\dfrac{\varepsilon_\omega}{hd^{k-1}}\right) & \text{otherwise} \end{cases}$$

$$\frac{\partial^k e_{\alpha 31}}{\partial x_1^{k_1}\partial x_2^{k_2}} = \mathcal{O}\left(\frac{h}{d^{k+1}}\varepsilon_1\right) \tag{0.11}$$

$$\frac{\partial^k e_{\alpha 32}}{\partial x_1^{k_1}\partial x_2^{k_2}\partial x_3^{k_3}} = \mathcal{O}\left(\frac{h^{3-k_3}}{d^{k+3-k_3}}\varepsilon_2\right) \quad \text{for} \quad k_3 = 0,1,2$$

$$\frac{\partial^k e_{ij2}}{\partial x_1^{k_1}\partial x_2^{k_2}\partial x_3^{k_3}} = \mathcal{O}\left(\frac{h\varepsilon_2}{d^{k+1}}\right) \begin{cases} \text{for } k_3 \text{ odd and } (i,j)= (\alpha,\beta)\text{or}(\ 3,3) \\ \text{for } k_3 \text{ even and } (i,j)=(\alpha,3) \end{cases}$$

The paragraph after (0.10) also applies to (0.11).

Using (0.10) and (0.11), expressions are obtained for the stresses and strains in terms of $\sigma_{\alpha\beta}$ and $b_{\alpha\beta}$. Those for the strains are (0.14) and those for the stresses are (0.15). To define the constants which appear in (0.14) and (0.15) we first introduce the strain invariants s_1, s_2, s_3 by

$$s_1 = e_{ii}\ , \qquad s_2 = e_{ij}e_{ji}\ , \qquad s_3 = e_{ij}e_{jk}e_{ki}. \tag{0.12}$$

We consider a material for which W is an analytic function of s_1, s_2, s_3 and define

$$\lambda = \left.\frac{\partial^2 W}{\partial s_1^2}\right|_{s_1=s_2=s_3=0}$$

$$\mu = \left.\frac{\partial W}{\partial s_2}\right|_{s_1=s_2=s_3=0}$$

$$W_{10} = \frac{1}{E}\left.\frac{\partial W}{\partial s_3}\right|_{s_1=s_2=s_3=0} \tag{0.13}$$

$$W_{20} = \frac{1}{E}\left.\frac{\partial^2 W}{\partial s_1 \partial s_2}\right|_{s_1=s_2=s_3=0}$$

$$W_{40} = \frac{1}{6E}\left.\frac{\partial^3 W}{\partial s_1^3}\right|_{s_1=s_2=s_3=0}$$

Many combinations of these five eleastic constants appear in formulas(0.14) through (0.17). They are denoted by the letter L with a subscript and are defined in the Appendix.

$$\begin{aligned}
e_{\alpha\beta 1} &= x_3 b_{\alpha\beta} - \frac{3\lambda + 4\mu}{6(\lambda+2\mu)} x_3^3\, b_{\gamma\gamma,\alpha\beta} + \mathcal{O}(\theta^4 \varepsilon_1)\\
e_{\alpha\beta 2} &= \sigma_{\alpha\beta} + \frac{x_3^2}{2}\left[\frac{\lambda}{\lambda+2\mu}(\sigma_{\gamma\gamma,\alpha\beta} - b_{\alpha\beta} b_{\gamma\gamma}) + b_{\alpha\gamma} b_{\gamma\beta}\right] + \mathcal{O}(\theta^4 \varepsilon_2)\\
e_{\alpha 31} &= \frac{\lambda+\mu}{\lambda+2\mu}(h^2 - x_3^2)\, b_{\gamma\gamma,\alpha} + \mathcal{O}(\theta^3 \varepsilon_1)\\
e_{\alpha 32} &= \frac{1}{6} x_3 (x_3^2 - h^2)\left(\frac{E}{2\mu} L_{10} b_{\gamma\nu} b_{\gamma\nu,\alpha} + L_{17} b_{\gamma\gamma} b_{\nu\nu,\alpha} + L_{18} b_{\alpha\gamma} b_{\nu\nu,\gamma}\right) + \mathcal{O}(\theta^5 \varepsilon_2)\\
e_{331} &= -\frac{\lambda}{\lambda+2\mu} x_3 b_{\gamma\gamma} + x_3\left[L_{14} + \frac{3\lambda+2\mu}{4(\lambda+\mu)} \frac{x_3^2}{h^2}\right]\sigma_{\gamma\nu} b_{\gamma\nu}\\
&\quad + x_3\left[L_{15} + \frac{\lambda(3\lambda+2\mu)}{4(\lambda+\mu)(\lambda+2\mu)} \frac{x_3^2}{h^2}\right]\sigma_{\gamma\gamma} b_{\nu\nu} + \mathcal{O}(\theta^4 \varepsilon_1)\\
e_{332} &= -\frac{\lambda}{\lambda+2\mu}\sigma_{\gamma\gamma} - \frac{EW_{20}}{\lambda+2\mu}\sigma_{\gamma\nu}\sigma_{\gamma\nu} - \frac{EL_3}{\lambda+2\mu}\sigma_{\gamma\gamma}\sigma_{\nu\nu} + \left(\frac{x_3^2}{2} L_4 - \frac{\mu h^2}{\lambda+2\mu}\right) b_{\gamma\nu} b_{\gamma\nu}\\
&\quad + \left[\frac{x_3^2}{2} L_5 - \frac{\lambda\mu\, h^2}{(\lambda+2\mu)^2}\right] b_{\gamma\gamma} b_{\nu\nu} + \mathcal{O}(0^4 \varepsilon_2)
\end{aligned} \tag{0.14}$$

$$t_{\alpha\beta 1}= x_3\delta_{\alpha\beta}\{\frac{2\lambda\mu}{E(\lambda+2\mu)}\ b_{\gamma\gamma}+[\frac{1}{2}\ L_8-\ \frac{\lambda\mu}{2E(\lambda+\mu)}\ \frac{x_3^2}{h^2}]\sigma_{\gamma\nu}b_{\gamma\nu}$$

$$+\ [L_{16}\ -\ \frac{\lambda^2\mu}{2E(\lambda+\mu)(\lambda+2\mu)}\ \frac{x_3^2}{h^2}]\sigma_{\gamma\gamma}b_{\nu\nu}\}$$

$$+x_3\{\frac{2\mu}{E}\ b_{\alpha\beta}-\ \frac{\mu(3\lambda+4\mu)}{3E(\lambda+2\mu)}\ x_3^2 b_{\gamma\gamma,\alpha\beta}+L_2(\sigma_{\alpha\beta}b_{\gamma\gamma}+\sigma_{\gamma\gamma}b_{\alpha\beta})$$

$$+3w_{10}(\sigma_{\alpha\gamma}b_{\gamma\beta}+\sigma_{\beta\gamma}b_{\gamma\alpha})\}\ +\ \acute{}\,(\theta^4\varepsilon_1)$$

$$t_{\alpha\beta 2}=\ \delta_{\alpha\beta}\{\frac{2\lambda\mu}{E(\lambda+2\mu)}\ \sigma_{\gamma\gamma}+\ \frac{2\mu W_{20}}{\lambda+2\mu}\ \sigma_{\gamma\nu}\sigma_{\gamma\nu}\ +\ L_{13}\sigma_{\gamma\gamma}\sigma_{\nu\nu}$$

$$+[\frac{x_3^2}{2}\ L_6-\ \frac{\lambda\mu}{E(\lambda+2\mu)}\ h^2]b_{\gamma\nu}b_{\gamma\nu}+[\frac{x_3^2}{2}\ L_7-\ \frac{\lambda^2\mu}{E(\lambda+2\mu)\,2}\ h^2]b_{\gamma\gamma}b_{\nu\nu}\}$$

$$+\ \frac{2\mu}{E}\ \sigma_{\alpha\beta}\ +\ L_2\sigma_{\alpha\beta}\sigma_{\gamma\gamma}\ +\ 3W_{10}\sigma_{\alpha\gamma}\sigma_{\gamma\beta}$$

$$+\ \frac{x_3^2}{2}\ [\frac{2\lambda\mu}{E(\lambda+2\mu)}\ \sigma_{\gamma\gamma,\alpha\beta}\ +\ L_8 b_{\alpha\beta}b_{\gamma\gamma}+\ L_9 b_{\alpha\gamma}b_{\gamma\beta}]\ +\ \ (\theta^4\varepsilon_2)$$

$$t_{\alpha 31}=\ (h^2-x_3^2)\{\frac{2\mu(\lambda+\mu)}{E(\lambda+2\mu)}\ b_{\gamma\gamma,\alpha}-\ \frac{1}{2}\ L_{30}\sigma_{\alpha\nu}b_{\gamma\gamma,\nu}$$

$$-\ \frac{1}{2}\ L_{31}\sigma_{\nu\nu,\gamma}b_{\gamma\alpha}-\ \frac{1}{2}\ L_{32}\ \sigma_{\alpha\gamma,\nu}\ b_{\gamma\nu}$$

$$+\ (L_{33}-\ \frac{\mu}{2E}\ \frac{x_3^2}{h^2})\sigma_{\gamma\nu,\alpha}b_{\gamma\nu}\ +\ (L_{34}-\ \frac{\mu}{2E}\ \frac{x_3^2}{h^2})\sigma_{\gamma\nu}b_{\gamma\nu,\alpha}$$

$$+[L_{35}-\ \frac{\lambda\mu}{2E(\lambda+2\mu)}\ \frac{x_3^2}{h^2}]\sigma_{\gamma\gamma,\alpha}b_{\nu\nu}+[L_{36}-\frac{\lambda\mu}{2E(\lambda+2\mu)}\ \frac{x_3^2}{h^2}]\sigma_{\gamma\gamma}b_{\nu\nu,\alpha}\}$$

$$+\ \acute{}\,(\theta^5\varepsilon_1) \tag{0.15}$$

$$t_{\ \alpha 32}=\ \frac{x_3}{6}\ (x_3^2-h^2)(L_{10}b_{\gamma\nu}b_{\gamma\nu,\alpha}+L_{11}b_{\gamma\gamma}b_{\nu\nu,\alpha}+L_{12}b_{\alpha\gamma}b_{\nu\nu,\gamma})+\acute{}\,(\theta^5\varepsilon_2)$$

$$t_{\ 331}=\ x_3(h^2-x_3^2)\{-\ \frac{2\mu(\lambda+\mu)}{3E(\lambda+2\mu)}\ b_{\gamma\gamma,\nu\nu}+$$

$$+\ \frac{1}{h^2}[L_{57}+\ \frac{3\lambda\mu}{20E(\lambda+\mu)}\ \frac{x_3^2}{h^2}]\sigma_{\omega\omega}(\sigma_{\gamma\nu}b_{\gamma\nu}+\ \frac{\lambda}{\lambda+2\mu}\ \sigma_{\gamma\gamma}b_{\nu\nu})$$

$$+[L_{58}-\ \frac{\mu(3\lambda+2\mu)}{10E(\lambda+2\mu)}\ \frac{x_3^2}{h^2}]\sigma_{\omega\omega,\gamma\nu}b_{\gamma\nu}+(L_{59}+\ \frac{\mu}{10E}\ \frac{x_3^2}{h^2})\sigma_{\gamma\nu}b_{\omega\omega,\gamma\nu}$$

$$+(L_{60}+\ \frac{\mu}{5E}\ \frac{x_3^2}{h^2})\sigma_{\gamma\nu,\omega}b_{\gamma\nu,\omega}+[L_{61}+\ \frac{\lambda\mu}{5E(\lambda+2\mu)}\ \frac{x_3^2}{h^2}]\sigma_{\gamma\gamma,\omega}b_{\nu\nu,\omega}$$

$$+\left(L_{62}+\frac{\mu}{10E}\frac{x_3^2}{h^2}\right)b_{\gamma\nu}b_{\nu\omega}b_{\omega\gamma}+\left[L_{63}-\frac{\mu(3\lambda+4\mu)}{40E(\lambda+\mu)}\frac{x_3^2}{h^2}\right]b_{\gamma\nu}b_{\gamma\nu}b_{\omega\omega}$$

$$+\left[L_{64}-\frac{\lambda\mu}{40E(\lambda+\mu)}\frac{x_3^2}{h^2}\right]b_{\gamma\gamma}b_{\nu\nu}b_{\omega\omega}\} + \mathcal{O}(\theta^6\varepsilon_1)$$

$$t_{332} = (x_3^2-h^2)\,\frac{\mu}{E}\,\left(b_{\gamma\nu}b_{\gamma\nu} + \frac{\lambda}{\lambda+2\mu}\,b_{\gamma\gamma}b_{\nu\nu}\right) + \mathcal{O}(\theta^4\varepsilon_2)\ .$$

The relations in (0.14) and (0.15) remain valid if they are differentiated any number of times with respect to x_1 or x_2 provided that the remainder term is multiplied by θ/h for each differentiation and provided that sufficiently many continuous derivatives exist. They also remain valid if they are differentiated with respect to x_3 provided that the remainder term is divided by an h for each differentiation. The constant multiplier in the remainder term will also generally need to be changed for each differentiation.

In determining the order of individual terms in (0.14) and (0.15) one needs estimates on $\sigma_{\alpha\beta}$, $b_{\alpha\beta}$, and their derivatives. Such results are

$$\frac{\partial^k\sigma_{\alpha\beta}}{\partial x_1^{k_1}\partial x_2^{k_2}} = \mathcal{O}\left(\frac{\varepsilon_2}{d^k}\right) \tag{0.16}$$

$$\frac{\partial^k b_{\alpha\beta}}{\partial x_1^{k_1}\partial x_2^{k_2}} = \mathcal{O}\left(\frac{\varepsilon_1}{hd^k}\right)$$

Finally the approximate differential equations for $\sigma_{\alpha\beta}$ and $b_{\alpha\beta}$ are

$$b_{\alpha\gamma,\beta}-b_{\alpha\beta,\gamma}=(\sigma_{\beta\nu,\alpha}+\sigma_{\alpha\nu,\beta}-\sigma_{\alpha\beta,\nu})b_{\gamma\nu}-(\sigma_{\gamma\nu,\alpha}+\sigma_{\alpha\nu,\gamma}-\sigma_{\alpha\gamma,\nu})b_{\beta\nu}$$

$$+\frac{\lambda}{\lambda+2\mu}(b_{\alpha\beta}\sigma_{\nu\nu,\gamma}-b_{\alpha\gamma}\sigma_{\nu\nu,\beta})+O(\frac{\theta^5}{h^2}\varepsilon_1)$$

$$\frac{\lambda}{\lambda+2\mu}\sigma_{\gamma\gamma,\alpha}+\sigma_{\alpha\gamma,\gamma}+L_{19}\sigma_{\gamma\nu}\sigma_{\gamma\nu,\alpha}+L_{20}\sigma_{\gamma\gamma}\sigma_{\nu\nu,\alpha}$$

$$+L_{21}\sigma_{\alpha\gamma}\sigma_{\nu\nu,\gamma}+L_{22}\sigma_{\alpha\gamma,\nu}\sigma_{\gamma\nu}$$

$$+h^2(L_{23}b_{\gamma\nu}b_{\gamma\nu,\alpha}+L_{24}b_{\gamma\gamma}b_{\nu\nu,\alpha}+L_{25}b_{\alpha\gamma}b_{\nu\nu,\gamma})=O(\frac{\theta^5}{h}\varepsilon_2)$$

$$\sigma_{\gamma\gamma,\nu\nu}=\frac{\lambda+2\mu}{4(\lambda+\mu)}(b_{\gamma\nu}b_{\gamma\nu}-b_{\gamma\gamma}b_{\nu\nu})+L_{76}\sigma_{\gamma\gamma,\nu\omega}\sigma_{\nu\omega}$$

$$+L_{77}\sigma_{\gamma\nu,\omega}\sigma_{\gamma\nu,\omega}+L_{78}\sigma_{\gamma\gamma,\omega}\sigma_{\nu\nu,\omega}+L_{79}\sigma_{\gamma\nu,\omega}\sigma_{\gamma\omega,\nu}$$

$$(0.17)\qquad +L_{80}\sigma_{\gamma\nu}b_{\nu\omega}b_{\omega\gamma}+L_{81}\sigma_{\gamma\nu}b_{\gamma\nu}b_{\omega\omega}+L_{82}\sigma_{\omega\omega}b_{\gamma\nu}b_{\gamma\nu}+L_{83}\sigma_{\gamma\gamma}b_{\nu\nu}b_{\omega\omega}$$

$$+h^2(L_{84}b_{\omega\omega,\gamma\nu}b_{\gamma\nu}+L_{85}b_{\gamma\nu,\omega}b_{\gamma\nu,\omega}+L_{86}b_{\gamma\gamma,\omega}b_{\nu\nu,\omega})+O(\frac{\theta^6}{h^2}\varepsilon_2)$$

$$b_{\gamma\gamma,\nu\nu}=\frac{3(\lambda+2\mu)}{2(\lambda+\mu)}\frac{1}{h^2}(\sigma_{\gamma\nu}b_{\gamma\nu}+\frac{\lambda}{\lambda+2\mu}\sigma_{\gamma\gamma}b_{\nu\nu})$$

$$+\frac{1}{h^2}(L_{65}\sigma_{\gamma\omega}\sigma_{\omega\nu}b_{\nu\gamma}+L_{66}\sigma_{\gamma\nu}\sigma_{\gamma\nu}b_{\omega\omega}$$

$$+L_{67}\sigma_{\omega\omega}\sigma_{\gamma\nu}b_{\gamma\nu}+L_{68}\sigma_{\gamma\gamma}\sigma_{\nu\nu}b_{\omega\omega})$$

$$+L_{69}\sigma_{\omega\omega,\gamma\nu}b_{\gamma\nu}+L_{70}\sigma_{\gamma\nu}b_{\omega\omega,\gamma\nu}+L_{71}\sigma_{\gamma\nu,\omega}b_{\gamma\nu,\omega}$$

$$+L_{72}\sigma_{\gamma\gamma,\omega}b_{\nu\nu,\omega}+L_{73}b_{\gamma\nu}b_{\nu\omega}b_{\omega\gamma}$$

$$+L_{74}b_{\gamma\nu}b_{\gamma\nu}b_{\omega\omega}+L_{75}b_{\gamma\gamma}b_{\nu\nu}b_{\omega\omega}+O(\frac{\theta^6}{h^3}\varepsilon_1)\ .$$

(0.16) and (0.17) are valid at all particles on the middle surface which were in a circle on the undeformed middle surface with center at $(\hat{x}_1,\hat{x}_2,0)$ and a radius depending on the strain energy density function.

If $\varepsilon_1 \le h^3/D^3$ and $\max |e_{ij2}| \le \frac{D^2}{h^2}\,\varepsilon_1^2$, then $\varepsilon_2 = \frac{D^2}{h^2}\,\varepsilon_1^2 \le \frac{h^4}{D^4}$, $\theta = \frac{h}{D}$, and $d = \theta_0 D$. Using these with (0.16), we see that (0.17) becomes

$$
\begin{aligned}
& b_{\alpha\gamma,\beta} - b_{\alpha\beta,\gamma} = \mathcal{O}\left(\frac{h^3}{D^5}\,\varepsilon_1\right) \\
& \frac{\lambda}{\lambda+2\mu}\,\sigma_{\gamma\gamma,\alpha} + \sigma_{\alpha\gamma,\gamma} \\
& \quad + h^2\left(L_{23} b_{\gamma\nu} b_{\gamma\nu,\alpha} + L_{24} b_{\gamma\gamma} b_{\nu\nu,\alpha} + L_{25} b_{\alpha\gamma} b_{\nu\nu,\gamma}\right) = \mathcal{O}\left(\frac{h^2}{D^3}\,\varepsilon_1^2\right) \\
& \sigma_{\gamma\gamma,\nu\nu} = \frac{\lambda+2\mu}{4(\lambda+\mu)}\left(b_{\gamma\nu} b_{\gamma\nu} - b_{\gamma\gamma} b_{\nu\nu}\right) \\
& \quad + h^2\left(L_{84} b_{\omega\omega,\gamma\nu} b_{\gamma\nu} + L_{85} b_{\gamma\nu,\omega} b_{\gamma\nu,\omega} + L_{86} b_{\gamma\gamma,\omega} b_{\nu\nu,\omega}\right) + \mathcal{O}\left(\frac{h^2}{D^4}\,\varepsilon_1^2\right) \\
& b_{\gamma\gamma,\nu\nu} = \frac{3(\lambda+2\mu)}{2(\lambda+\mu)}\,\frac{1}{h^2}\left(\sigma_{\gamma\nu} b_{\gamma\nu} + \frac{\lambda}{\lambda+2\mu}\,\sigma_{\gamma\gamma} b_{\nu\nu}\right) + \mathcal{O}\left(\frac{h^3}{D^6}\,\varepsilon_1\right)
\end{aligned}
\tag{0.18}
$$

Similar modifications will also result in (0.14) and (0.15). If $\varepsilon_2 \le \frac{h^2}{D^2}$ and $\varepsilon_1 \le \frac{h^2}{D^2}\sqrt{\varepsilon_2}$, then $\theta = \frac{h}{D}$ and $d = \theta_0 D$. If we use these in (0.17), the first equation does not simplify, but the last three equations become

$$\begin{aligned}
&\frac{\lambda}{\lambda+2\mu}\,\sigma_{\gamma\gamma,\alpha}+\sigma_{\alpha\gamma,\gamma}+L_{19}\sigma_{\gamma\nu}\sigma_{\gamma\nu,\alpha}+L_{20}\sigma_{\gamma\gamma}\sigma_{\nu\nu,\alpha}\\
&\quad+L_{21}\sigma_{\alpha\gamma}\sigma_{\nu\nu,\gamma}+L_{22}\sigma_{\alpha\gamma,\nu}\sigma_{\gamma\nu}=O\left(\frac{h^4}{D^5}\,\varepsilon_2\right)\\
&\sigma_{\gamma\gamma,\nu\nu}=\frac{\lambda+2\mu}{4(\lambda+\mu)}\left(b_{\gamma\nu}b_{\gamma\nu}-b_{\gamma\gamma}b_{\nu\nu}\right)+L_{76}\sigma_{\gamma\gamma,\nu\omega}\,\sigma_{\nu\omega}\\
&\quad+L_{77}\sigma_{\gamma\nu,\omega}\sigma_{\gamma\nu,\omega}+L_{78}\sigma_{\gamma\gamma,\omega}\sigma_{\nu\nu,\omega}+L_{79}\sigma_{\gamma\nu,\omega}\sigma_{\gamma\omega,\nu}+O\left(\frac{h^4}{D^6}\,\varepsilon_2\right)\\
&b_{\gamma\gamma,\nu\nu}=\frac{3(\lambda+2\mu)}{2(\lambda+\mu)}\,\frac{1}{h^2}\left(\sigma_{\gamma\nu}b_{\gamma\nu}+\frac{\lambda}{\lambda+2\mu}\,\sigma_{\gamma\gamma}b_{\nu\nu}\right)\\
&\quad+\frac{1}{h^2}\left(L_{65}\sigma_{\gamma\omega}\sigma_{\omega\nu}b_{\nu\gamma}+L_{66}\sigma_{\gamma\nu}\sigma_{\gamma\nu}b_{\omega\omega}+L_{67}\sigma_{\omega\omega}\sigma_{\gamma\nu}b_{\gamma\nu}+L_{68}\sigma_{\gamma\gamma}\sigma_{\nu\nu}b_{\omega\omega}\right)\\
&\quad+L_{69}\sigma_{\omega\omega,\gamma\nu}b_{\gamma\nu}+L_{70}\sigma_{\gamma\nu}b_{\omega\omega,\gamma\nu}+L_{71}\sigma_{\gamma\nu,\omega}b_{\gamma\nu,\omega}+L_{72}\sigma_{\gamma\gamma,\omega}b_{\nu\nu,\omega}\\
&\quad+O\left(\frac{h^3}{D^6}\,\varepsilon_1\right)
\end{aligned}\tag{0.19}$$

Various other simplifications are possible.

As a buckled plate approaches the unbuckled but strained configuration, we have $\varepsilon_1 \to 0$ and $b_{\alpha\beta} \to 0$ provided that the horizontal displacements are even functions of x_3 and the vertical displacement is an odd function of x_3 in the unbuckled configuration. This is the usual situation. Consequently, we see that all terms in equations 1 and 4 of (0.17) tend to zero as the plate unbuckles. This is an imporvement over the corresponding results obtained in [1] where the one strain parameter approaches a positive limit as the plate unbuckles and consequently, in some equations the error estimate does not go to zero but all other terms do. It has not yet been determined whether similar vacuities can occur in the results presented here.

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1. Formulation of the Theory

Since the strain energy density function W is an analytic function of the invariants s_1, s_2, s_3, we can write

$$(1.1) \quad W = \frac{\lambda}{2} s_1^2 + \mu s_2 + W_1 s_3 + W_2 s_1 s_2 + W_3 s_2^2 + W_4 s_1^3$$

where λ and μ are the Lame constants defined in (0.13) and W_1, W_2, W_3, W_4 are analytic functions of s_1, s_2, s_3.

From (0.3) it follows that

$$(1.2) \quad t_{ij} = \frac{1}{E} \left(\frac{\partial W}{\partial s_1} \delta_{ij} + 2 \frac{\partial W}{\partial s_2} e_{ij} + 3 \frac{\partial W}{\partial s_3} e_{ik} e_{kj}\right) .$$

From (0.5) we obtain the following in which the first line is a definition:

$$(1.3) \quad \begin{aligned} p_{ij} &= 2 e_{ik} t_{kj} - \frac{W}{E} \delta_{ij} \\ t_{ij,j} + p_{ij,j} &= 0 \end{aligned}$$

Making the definitions

$$(1.4) \quad \begin{aligned} \Gamma_{ijk} &= e_{ik,i} + e_{ik,j} - e_{ij,k} \\ (g^{ij}) &= (\delta_{ij} + 2 e_{ij})^{-1} \\ c_{ijk\ell} &= \Gamma_{ijn} g^{nm} \Gamma_{k\ell m} - \Gamma_{ikn} g^{nm} \Gamma_{j\ell m} \end{aligned}$$

the compatibility conditions are

$$(1.5) \quad e_{j\ell,ik} + e_{ik,j\ell} - e_{ij,k\ell} - e_{k\ell,ij} = c_{ijk\ell} .$$

(1.1) and (1.2) yield

$$
\text{(1.6)} \quad
\begin{aligned}
t_{ij} &= \frac{1}{E}\,(\lambda e_{kk}\delta_{ij} + 2\mu e_{ij}) + z_{ij} \\
z_{ij} &= \frac{1}{E}\,\Big[\Big(W_2 s_2 + 3W_4 s_1^2 + s_3\,\frac{\partial W_1}{\partial s_1} + s_1 s_2\,\frac{\partial W_2}{\partial s_1} + s_2^2\,\frac{\partial W_3}{\partial s_1} + s_1^3\,\frac{\partial W_4}{\partial s_1}\Big)\delta_{ij} \\
&\quad + 2e_{ij}\Big(W_2 s_1 + 2W_3 s_2 + s_3\,\frac{\partial W_1}{\partial s_2} + s_1 s_2\,\frac{\partial W_2}{\partial s_2} + s_2^2\,\frac{\partial W_3}{\partial s_2} + s_1^3\,\frac{\partial W_4}{\partial s_2}\Big) \\
&\quad + 3e_{ik}e_{kj}\Big(W_1 + s_3\,\frac{\partial W_1}{\partial s_3} + s_1 s_2\,\frac{\partial W_2}{\partial s_3} + s_2^2\,\frac{\partial W_3}{\partial s_3} + s_1^3\,\frac{\partial W_4}{\partial s_3}\Big)\Big]
\end{aligned}
$$

The first equation in (1.6) can be inverted when the strains are near zero (i.e. when θ_0 is sufficiently small) giving

$$
\text{(1.7)} \qquad e_{ij} = \frac{E}{2\mu}\,\Big(t_{ij} - \frac{\lambda}{3\lambda+2\mu}\,t_{kk}\delta_{ij}\Big) + r_{ij}
$$

where r_{ij} is quadratic in the stresses with coefficients depending on the stresses.

Letting $T = t_{kk}$, the compatibility conditions (1.5) can be expressed in terms of the stresses as

$$
\text{(1.8)} \quad
\begin{aligned}
& t_{j\ell,ik} + t_{ik,j\ell} - t_{ij,k\ell} - t_{k\ell,ij} \\
& - \frac{\lambda}{3\lambda+2\mu}\,(T_{,ik}\delta_{j\ell} + T_{,j\ell}\delta_{ik} - T_{,k\ell}\delta_{ij} - T_{,ij}\delta_{k\ell}) \\
& = \frac{2\mu}{E}\,c_{ijk\ell} - \frac{2\mu}{E}\,(r_{j\ell,ik} + r_{ik,j\ell} - r_{ij,k\ell} - r_{k\ell,ij})
\end{aligned}
$$

Functions $p_{ij\omega}$, $z_{ij\omega}$, $r_{ij\omega}$, T_ω, g^{ij}_{ω} , $\Gamma_{ijk\omega}$, $c_{ijk\ell\omega}$ can be introduced in the same way that $t_{ij\omega}$ and $e_{ij\omega}$ were introduced, e.g. $c_{ijk\ell} = c_{ijk\ell 1} + c_{ijk\ell 2}$ where $c_{ijk\ell\omega}$ is an even function of x_3 if ω plus the number of threes among i,j,k,ℓ is even. Otherwise $c_{ijk\ell\omega}$ is odd. Also T_1 is the odd part of T and T_2 is the even part. Splitting some of the preceding relationships into odd and even parts, we obtain

$$t_{ij\omega,j} + p_{ij\omega,j} = 0$$

$$t_{i3\omega} = 0 \quad \text{for} \quad x_3 = \pm h$$

$$e_{j\ell\omega,ik} + e_{ik\omega,j\ell} - e_{ij\omega,k\ell} - e_{k\ell\omega,ij} = c_{ijk\ell\omega}$$

$$t_{ij\omega} = \frac{1}{E}(\lambda e_{kk\omega}\delta_{ij} + 2\mu e_{ij\omega}) + z_{ij\omega}$$

(1.9)

$$e_{ij\omega} = \frac{E}{2\mu}\left(t_{ij\omega} - \frac{\lambda}{3\lambda+2\mu} T_\omega \delta_{ij}\right) + r_{ij\omega}$$

$$\begin{aligned} & t_{j\ell\omega,ik} + t_{ik\omega,j\ell} - t_{ij\omega,k\ell} - t_{k\ell\omega,ij} \\ & - \frac{\lambda}{3\lambda+2\mu}(T_{\omega,ik}\delta_{j\ell} + T_{\omega,j\ell}\delta_{ik} - T_{\omega,k\ell}\delta_{ij} - T_{\omega,ij}\delta_{k\ell}) \\ & = \frac{2\mu}{E} c_{ijk\ell\omega} - \frac{2\mu}{E}(r_{j\ell\omega,ik} + r_{ik\omega,j\ell} - r_{ij\omega,k\ell} - r_{k\ell\omega,ij}) \end{aligned}$$

It is easy to write equations (1.9) from the corresponding previous relations since they are linear in the symbols exhibited. Next we split nonlinear equations such as (1.6) into even and odd parts. This will be accomplished through several steps.

Let t_ω, e_ω, p_ω, c_ω, z_ω, r_ω (ω=1,2) denote the sets of functions $\{t_{ij\omega}\}$, $\{e_{ij\omega}\}$, $\{p_{ij\omega}\}$, $\{c_{ijk\ell\omega}\}$, $\{z_{ij\omega}\}$, $\{r_{ij\omega}\}$ respectively.

Let $a_1, a_2, \ldots, a_k$ denote any sets of functions. Then $a_1^{n_1} a_2^{n_2} \ldots a_k^{n_k} F(a_1, a_2, \ldots, a_k)$ will denote a function which is a sum whose terms are the product of n_1 elements of a_1, n_2 elements of a_2, ..., n_k elements of a_k, and a coefficient which is a function of the elements of $a_1, a_2, \ldots, a_k$. We will use notation such as $(a_1^{n_1} a_2^{n_2} + a_1^{\bar{n}_1} a_2^{\bar{n}_2}) F(a_1, a_2)$ for $a_1^{n_1} a_2^{n_2} F(a_1, a_2) + a_1^{\bar{n}_1} a_2^{\bar{n}_2} F(a_1, a_2)$. Notice that meaning has not been given to F itself so it is meaningless to ask if F is the same wherever it appears.

We can write $s_i = s_{ie} + s_{i\phi}$ where s_{ie} is an even function of x_3 and $s_{i\phi}$ is an odd function of x_3. From (0.12) we obtain

$$
\begin{aligned}
s_{1e} &= e_{ii2} \\
s_{1\phi} &= e_{ii1} \\
s_{2e} &= e_{ij1}e_{ij1} + e_{ij2}e_{ij2} \\
s_{2\phi} &= 2e_{ij1}e_{ij2} \\
s_{3e} &= 3e_{ij1}e_{jk1}e_{ki2} + e_{ij2}e_{jk2}e_{ki2} \\
s_{3\phi} &= e_{ij1}e_{jk1}e_{ki1} + 3e_{ij2}e_{jk1}e_{ki2}
\end{aligned}
\tag{1.10}
$$

Following the above procedure, the subscript e will always denote the even part and the subscript ϕ will always denote the odd part.

Since $s_{i\phi} = e_1F(e_1,e_2)$ (i=1,2,3), the same will also be true for any analytic function of s_1, s_2,s_3. Thus, for example, $W_{1\phi} = e_1F(e_1,e_2)$.

From (1.6)

$$
\begin{aligned}
z_{ij1} &= (e_1^2+e_1e_2)F(e_1,e_2) \\
z_{ij2} &= (e_1^2 + e_2^2)\ F(e_1,e_2) \\
t_{ij1} &= \frac{1}{E}\ (\lambda e_{kk1}\delta_{ij} + 2\mu e_{ij1}) + (e_1^2+e_1e_2)F(e_1,e_2) \\
t_{ij2} &= \frac{1}{E}\ (\lambda e_{kk2}\delta_{ij} + 2\mu e_{ij2}) + (e_1^2+e_2^2)\ F(e_1,e_2)
\end{aligned}
\tag{1.11}
$$

The last two lines can be inverted when the strains are sufficiently near zero (i.e. when θ_0 is sufficiently small) to obtain

$$\begin{aligned} e_{ij1} &= t_1\, F(t_1,t_2) \\ e_{ij2} &= (t_2+t_1^2)\, F(t_1,t_2) \end{aligned} \tag{1.12}$$

Substituting into (1.11) for $z_{ij\omega}$, we have

$$\begin{aligned} z_{ij1} &= (t_1^2+t_1t_2)F(t_1,t_2) \\ z_{ij2} &= (t_1^2+t_2^2)\, F(t_1,t_2) \end{aligned} \tag{1.13}$$

From $(1.9)_4$,

$$T_\omega = \frac{3\lambda+2\mu}{E}\, e_{kk\omega} + z_{kk\omega}$$

and

$$e_{ij\omega} = \frac{E}{2\mu}\, [t_{ij\omega} - \lambda\delta_{ij}(\frac{T_\omega - z_{kk\omega}}{3\lambda+2\mu}) - z_{ij\omega}] \; .$$

Therefore from $(1.9)_5$,

$$r_{ij\omega} = \frac{\lambda E}{2\mu(3\lambda+2\mu)}\, z_{kk\omega}\delta_{ij} - \frac{E}{2\mu}\, z_{ij\omega}$$

and from (1.13)

$$\begin{aligned} r_{ij1} &= (t_1^2 + t_1t_2)\, F(t_1,t_2) \\ r_{ij2} &= (t_1^2 + t_2^2)\, F(t_1,t_2) \end{aligned} \tag{1.14}$$

From (1.1) and (1.12),

$$\frac{1}{E}\, W_e = (e_1^2 + e_2^2)\, F(e_1,e_2) = (t_1^2+t_2^2)\, F(t_1,t_2)$$

$$\frac{1}{E}\, W_\phi = (e_1e_2+e_1^2)\, F(e_1,e_2) = (t_1^2+t_1t_2)\, F(t_1,t_2) \; .$$

From $(1.3)_1$,

$$p_{ij1} = 2e_{ik1}t_{kj2} + 2e_{ik2}t_{kj1} - \frac{1}{E}\, W_\phi\delta_{ij}$$

$$p_{ij2} = 2e_{ik1}t_{kj1} + 2e_{ik2}t_{kj2} - \frac{1}{E}\, W_e\delta_{ij} \; .$$

From these we obtain

$$p_{ij1} = (t_1t_2+t_1^2)\ F(t_1,t_2) \tag{1.15}$$

$$p_{ij2} = (t_1^2+t_2^2)\ F(t_1,t_2)$$

From $(1.4)_2$,

$$g^{ij} = \delta_{ij} - 2e_{ij} + 4e_{ik}e_{kj} - 8e_{ik}e_{k\ell}e_{\ell j} + \cdots \tag{1.16}$$

when the strains are sufficiently near zero. Again this holds when θ_0 is sufficiently small.

Using (1.12) we have

$$g_1^{ij} = e_1F(e_1,e_2) = t_1\ F(t_1,t_2) \tag{1.17}$$

$$g_2^{ij} = \delta_{ij} + F(e_1,e_2) = \delta_{ij} + F(t_1,t_2)$$

For any set A of functions of x_1, x_2, x_3 let A' be the set of all first order derivatives and let $\dot{A}$ be the set of all first order derivatives with respect to x_1 and x_2. Since $\dot{A}$ and A' are sets of functions, the preceding definition defines $\ddot{A}$, $\dot{A}'$, A" etc. We will also use the notation $A_{,3}$ for the set of all first order derivatives with respect to x_3 of functions in A.

From (1.4) we obtain

$$\Gamma_{ijk\omega} = e_{jk\omega,i} + e_{ik\omega,j} - e_{ij\omega,k}$$

$$\begin{aligned} c_{ijk\ell 1} = {} & \Gamma_{ijn1}(g^{nm}{}_2\ \Gamma_{k\ell m2} + g^{nm}{}_1\ \Gamma_{k\ell m1}) \\ & +\Gamma_{ijn2}(g^{nm}{}_1\ \Gamma_{k\ell m2} + g^{nm}{}_2\ \Gamma_{k\ell m1}) \\ & -\Gamma_{ikn1}(g^{nm}{}_2\ \Gamma_{j\ell m2} + g^{nm}{}_1\ \Gamma_{j\ell m1}) \\ & -\Gamma_{ikn2}(g^{nm}{}_1\ \Gamma_{j\ell m2} + g^{nm}{}_2\ \Gamma_{j\ell m1}) \end{aligned} \tag{1.18}$$

$$
\begin{aligned}
c_{ijk\ell 2} = \Gamma_{ijn1}&(g^{nm}{}_1\ \Gamma_{k\ell m2} + g^{nm}{}_2\ \Gamma_{k\ell m1})\\
+\Gamma_{ijn2}&(g^{nm}{}_1\ \Gamma_{k\ell m1} + g^{nm}{}_2\ \Gamma_{k\ell m2})\\
-\Gamma_{ikn1}&(g^{nm}{}_1\ \Gamma_{j\ell m2} + g^{nm}{}_2\ \Gamma_{j\ell m1})\\
-\Gamma_{ikn2}&(g^{nm}{}_1\ \Gamma_{j\ell m1} + g^{nm}{}_2\ \Gamma_{j\ell m2})
\end{aligned}
$$

From (1.18) and (1.17)

$$
(1.19)\qquad
\begin{aligned}
c_{ijk\ell 1} &= [(t_1')^2 + t_1't_2' + t_1(t_2')^2]\ F(t_1,t_2)\\
c_{ijk\ell 2} &= [(t_1')^2 + t_1't_2' + (t_2')^2]\ F(t_1,t_2)
\end{aligned}
$$

In (1.20) we have relisted some of the previous results. Each equation of (1.20) contains one or more sets of functions on the left-hand side, and the form of a function is shown on the right-hand side. The equations mean that any function in one of the sets on the left-hand side is of the form shown on the right-hand side.

$$
(1.20)\qquad
\begin{aligned}
p_1,r_1,z_1 &= t_1(t_1+t_2)\ F(t_1,t_2)\\
p_2,r_2,z_2 &= (t_1^2+t_2^2)\ F(t_1,t_2)\\
c_1 &= [t_1'(t_1'+t_2') + t_1t_2'^2]\ F(t_1,t_2)\\
c_2 &= (t_1'+t_2')^2\ F(t_1,t_2)
\end{aligned}
$$

2. Relations Between the Derivatives of the Stresses

Let $a_{k\omega}$ $(k=1,2,\ldots;\ \omega=1,2)$ be the set of functions of the type

$$\frac{\partial^k t_{ij\omega}}{\partial x_1^{k_1} \partial x_2^{k_2} \partial x_3^{k_3}}$$

such that $k_3=0$; or $k_3=1$ and $(i,j) = (\alpha,3)$ or $(3,3)$; or k_3 is even and $(i,j) = (\alpha,\beta)$ or $(3,3)$; or k_3 is odd and $(i,j)=(\alpha,3)$. The various cases are not mutually exclusive.

Let $b_{k\omega}$ $(k=1,2,\ldots;\ \omega=1,2)$ be the set of all functions

$$\frac{\partial^k t_{ij\omega}}{\partial x_1^{k_1} \partial x_2^{k_2} \partial x_3^{k_3}}$$

with no restrictions.

We next show that

$$\begin{aligned}
&a_{1\omega} = \dot{t}_\omega + p_\omega' \\
&b_{1\omega} = \{t_{\alpha\beta\omega,3}\} + \dot{t}_\omega + p_\omega' \\
(2.1)\qquad &a_{k\omega} = \dot{t}_\omega^{(k)} + p_\omega^{'(k)} + r_\omega^{'(k)} + c_\omega^{'(k-2)} \quad \text{for } k=2,3,\ldots \\
&b_{k\omega} = \{\dot{t}_{\alpha\beta\omega,3}^{(k-1)}\} + \dot{t}_\omega^{(k)} + p_\omega^{'(k)} + r_\omega^{'(k)} + c_\omega^{'(k-2)} \quad \text{for } k=2,3..
\end{aligned}$$

where $\dot{t}_\omega^{(k)}$ denotes t_ω with k superdots, $p_\omega^{'(k)}$ denotes p_ω with k super primes, $\{\dot{t}_{\alpha\beta\omega,3}^{(k-1)}\}$ denotes the set of all functions of the type

$$\frac{\partial^k t_{\alpha\beta\omega}}{\partial x_1^{k_1} \partial x_2^{k_2} \partial x_3}, \text{ etc.}$$

Also the notation $a_1+a_2+\ldots+a_k$, where $a_1,a_2,\ldots,a_k$ are sets of functions, denotes a function which is a linear combination of elements of $a_1,a_2,\ldots,a_k$ with constant coefficients depending only on the material.

The functions in $a_{1\omega}$ are those of the types $t_{ij\omega,\alpha}$ and $t_{i3\omega,3}$. The conclusion for functions in $a_{1\omega}$ follows immediately from

$$t_{i3\omega,3} = -t_{i\alpha\omega,\alpha} - p_{ij\omega,j} \tag{2.2}$$

which follows from (1.9) .

The only functions in $b_{1\omega}$ which are not in $a_{1\omega}$ are of the type $t_{\alpha\beta\omega,3}$. Thus the conclusions is trivial for $b_{1\omega}$.

Each function in $a_{2\omega}$ $(b_{2\omega})$ is the derivative with respect to x_1 or x_2 of a function in $a_{1\omega}$ $(b_{1\omega})$ unless it is of the form $t_{ij\omega,33}$. Then clearly (2.1) is valid for those functions of $a_{2\omega}$ and $b_{2\omega}$ for which not all differentiations are with respect to x_3.

From (2.2) we have $t_{i3\omega,33} = -t_{i\alpha\omega,\alpha 3} - p_{ij\omega,j3}$. We already know that (2.1) is valid for the functions $t_{i\alpha\omega,\alpha 3}$ so that $t_{\beta 3\omega,33} = \dot{t}_{\omega,3} + p''_{\omega}$ and $t_{33\omega,33} = \ddot{t}_{\omega} + p''_{\omega}$. This takes care of all functions in $a_{2\omega}$ and $b_{2\omega}$ with the exception of the functions $t_{\alpha\beta\omega,33}$ which are in both $a_{2\omega}$ and $b_{2\omega}$.

Setting $k = \ell = 3$ and $j = i$ and summing over i in $(1.9)_6$, we obtain

$$\begin{aligned} T_{\omega,33} = \frac{3\lambda+2\mu}{\lambda+2\mu}\,[2t_{i3\omega,i3} - t_{33\omega,ii} + \frac{\lambda}{3\lambda+2\mu}\,T_{\omega,\alpha\alpha} \\ - \frac{2\mu}{E}\,(c_{ii33\omega} - 2r_{i3\omega,i3} + r_{ii\omega,33} + r_{33\omega,ii})] \ . \end{aligned} \tag{2.3}$$

Setting $i = \alpha$, $j = \beta$, $k = \ell = 3$ in $(1.9)_6$, we obtain

$$\begin{aligned} t_{\alpha\beta\omega,33} = t_{\beta 3\omega,\alpha 3} + t_{\alpha 3\omega,\beta 3} - t_{33\omega,\alpha\beta} + \frac{\lambda}{3\lambda+2\mu}(T_{\omega,33}\delta_{\alpha\beta} + T_{\omega,\alpha\beta}) \\ - \frac{2\mu}{E}(c_{\alpha\beta 33\omega} - r_{\beta 3\omega,\alpha 3} - r_{\alpha 3\omega,\beta 3} + r_{\alpha\beta\omega,33} + r_{33\omega,\alpha\beta}) \end{aligned} \tag{2.4}$$

(2.3) tells us that $T_{\omega,33}$ is a linear combination of certain functions in $a_{2\omega}$ and functions of the types $c_{ijk\ell\omega}$ and $r_{ij\omega,k\ell}$. The functions from $a_{2\omega}$ involved are those for which we already know (2.1) holds. Thus $T_{\omega,33}$ is a linear combination of functions of the types $t_{ij\omega,\alpha\beta}$; $p_{ij\omega,k\ell}$; $r_{ij\omega,k\ell}$; $c_{ijk\ell\omega}$.

(2.4) tells us that $t_{\alpha\beta\omega,33}$ is a linear combination of certain functions of $a_{2\omega}$, of $T_{\omega,33}$, and of functions of the types $c_{ijk\ell\omega}$ and $r_{ij\omega,k\ell}$. The functions from $a_{2\omega}$ involved are those for which we already know (2.1) holds. Combining this with our information about $T_{\omega,33}$, we obtain (2.1) for $t_{\alpha\beta\omega,33}$. This completes the proof of (2.1) for $k = 2$.

For $k \geq 3$, (2.1) follows by induction in a similar manner. If it is true for some k, each function in $a_{(k+1)\omega}$ ($b_{(k+1)\omega}$) is the derivative with respect to x_1 or x_2 of a function in $a_{k\omega}$ ($b_{k\omega}$) unless it is of the form $\partial^{k+1}t_{ij\omega}/\partial x_3^{k+1}$. Thus (2.1) is valid for those functions of $a_{(k+1)\omega}$ and $b_{(k+1)\omega}$ for which not all differentiations are with respect to x_3. We then obtain (2.1) for functions of the type $\partial^{k+1}t_{ij\omega}/\partial x_3^{k+1}$ by differentiating (2.2), (2.3) and (2.4) with respect to x_3 the appropriate number of times.

Next we show that (2.1) leads to

$$
\begin{aligned}
a_{k1} &= \dot{t}_1^{(k)} + [t_1(t_1+t_2)\ F(t_1,t_2)]'^{(k)} \\
a_{k2} &= \dot{t}_2^{(k)} + [(t_1^2+t_2^2)\ F(t_1,t_2)]'^{(k)} \\
b_{k1} &= \{\dot{t}_{\alpha\beta1,3}^{(k-1)}\} + \dot{t}_1^{(k)} + [t_1(t_1+t_2)\ F(t_1,t_2)]'^{(k)} \\
b_{k2} &= \{\dot{t}_{\alpha\beta2,3}^{(k-1)}\} + \dot{t}_2^{(k)} + [(t_1^2+t_2^2)\ F(t_1,t_2)]'^{(k)}
\end{aligned}
\tag{2.5}
$$

which is valid for k=1,2,3,... .

The expression $[t_1(t_1+t_2)\ F(t_1,t_2)]'^{(k)}$ should not be interpreted as a kth derivative of a function of the type $t_1(t_1+t_2)\ F(t_1,t_2)$. It indicates a function of the type obtained if one differentiates formally. Thus $[t_1(t_1+t_2)\ F(t_1,t_2)]'$ indicates a function of the type $[(t_1+t_2)t_1' + t_1t_2']F(t_1,t_2)$ rather than the derivative of a function of the type $t_1(t_1+t_2)F(t_1,t_2)$. With this notation, for example, the third and fourth lines of (1.20) become

$$c_1 = [t_1(t_1+t_2)F(t_1,t_2)]''$$
$$c_2 = [(t_1^2+t_2^2)F(t_1,t_2)]'' \quad .$$

(2.5) now follows from the above, (1.20) and (2.1).

3. Proof of (0.10) and (0.11)

Let

$$r = \sqrt{(x_1-\hat{x}_1)^2+(x_2-\hat{x}_2)^2}$$

$$(3.1)\qquad \zeta_k(r) = \begin{cases} \left(1-\frac{r^2}{d^2}\right)^2 & \text{for } 0 \le r \le d \\ 0 & \text{for } r \ge d \end{cases} \quad \text{for } k = 0,1,2$$

$$\zeta_k(r) = \begin{cases} \left(1-\frac{2^{2k-4}r^2}{d^2}\right)^2 & \text{for } 0 \le r \le \frac{d}{2^{k-2}} \\ 0 & \text{for } r \ge \frac{d}{2^{k-2}} \end{cases} \quad \text{for } k = 3,4,\ldots$$

Then we have

$$(3.2)\qquad \begin{aligned} |\dot{\zeta}_k| &= \mathcal{O}\left(\frac{1}{d}\sqrt{\zeta}\right) = \mathcal{O}\left(\frac{1}{d}\right) \\ |\ddot{\zeta}_k| &= \mathcal{O}\left(\frac{1}{d^2}\right) . \end{aligned}$$

The notation $B = \mathcal{O}(C)$ will always mean there is a constant κ depending only on the material and on universal constants such as the value of k in (3.2) such that $|B| \le \kappa|C|$. In (3.2) the constant κ is actually independent of the material, however. Also for a finite set A of functions, let $|A|^2$ be the sum of the squares of the functions in A and let $|A| \ge 0$. This sdefines $|\dot{\zeta}_k|$ and $|\ddot{\zeta}_k|$ in (3.2).

Let $||A||_k^2 = \iiint |A|^2\, dx_1\, dx_2\, dx_3$ where A is a finite set of functions and the integration is over the set where $|x_3| \le h$ and $\zeta_k \ne 0$. We let

$$(3.3)\qquad A_{k\omega} = ||\zeta_k a_{k\omega}||_k , \quad B_{k\omega} = ||\zeta_k b_{k\omega}||_k \quad (k=1,2,\ldots;\ \omega=1,2).$$

If A is any finite set of functions, we have (see Appendix of first paper of [1])

$$
\begin{aligned}
&||\zeta_k\dot{A}||_k = \mathcal{O}(\tfrac{1}{d}\,||A||_k) + o(d||\zeta_k\ddot{A}||_k)\\
&||\zeta_k A_{,3}||_k = \mathcal{O}(\tfrac{1}{h}||A||_k) + o(h||\zeta_k A_{,33}||_k)\\
&||\zeta_k A'||_k = \mathcal{O}(\tfrac{1}{h}||A||_k) + o(d||\zeta_k\ddot{A}||_k + h||\zeta_k A_{,33}||_k)\\
&||\dot{\zeta}_k A_{,3}||_k = \mathcal{O}(\tfrac{1}{hd}\,||A||_k) + o(\tfrac{h}{d}\,||\zeta_k A_{,33}||_k)\\
&||\dot{\zeta}_k\dot{A}||_k = \mathcal{O}(\tfrac{1}{d^2}\,||A||_k) + o(||\zeta_k\ddot{A}||_k)\\
(3.4)\qquad &||\dot{\zeta}_k A'||_k = \mathcal{O}(\tfrac{1}{hd}\,||A||_k) + o(||\zeta_k A''||_k)\\
&||\zeta_k\dot{A}\,\dot{A}||_k = \mathcal{O}\,[\bar{A}(||\zeta_k\ddot{A}||_k + \tfrac{1}{d^2}\,||A||_k)]\\
&||\zeta_k A_{,3}A_{,3}||_k = \mathcal{O}\,[\bar{A}(||\zeta_k A_{,33}||_k + \tfrac{1}{h^2}\,||A||_k)]\\
&||\zeta_k A'A'||_k = \mathcal{O}\,[\bar{A}(||\zeta_k A''||_k + \tfrac{1}{h^2}\,||A||_k]\\
&||\zeta_k\dot{A}_{,3}||_k = \mathcal{O}(\tfrac{1}{hd}\,||A||_k + \tfrac{d}{h}\,||\zeta_k\ddot{A}||_k + \tfrac{h}{d}\,||\zeta_k A_{,33}||_k)\\
&\tfrac{\sqrt{h}}{d}\,\zeta_k|A| = \mathcal{O}\,[||\zeta_k\ddot{A}||_k + \tfrac{1}{d^2}\,||A||_k + \tfrac{h^2}{d^2}\,||\zeta_k A''||_k]
\end{aligned}
$$

where $\bar{A}$ is an upper bound for the absolute values of the functions in A.

The notation $B = \mathcal{O}(C) + o(D)$ means for each constant $k > 0$ there is a $\kappa > 0$ depending only on the material and on k such that $|B| \leq \kappa|C| + k|D|$. In (3.4) κ is independent of the material, but this will not be the case in later uses of this notation.

We now show that

$$||\zeta_k \overset{\bullet}{t}{}_{\omega}^{(k)}||_k = \mathcal{C}\,[\frac{1}{d^2}\, ||\overset{\bullet}{e}{}_{\omega}^{(k-2)}||_k + \frac{d}{h}\, ||\zeta_k p_{\omega}^{'\cdot(k-1)}||_k$$

$$(3.5) \qquad + ||\zeta_k \overset{\bullet}{c}{}_{\omega}^{(k-2)}||_k + ||\zeta_k z_{\omega}^{'(k)}||_k]$$

$$+ \, o[||\zeta_k \overset{\bullet}{e}{}_{\omega}^{(k)}||_k + \frac{h}{d}\, ||\zeta_k e_{\omega}^{'(k)}||_k]$$

for k = 2,3,... where $\overset{\bullet}{A}{}^{(k)}$ denotes A with k super dots, $A^{'\cdot(k-1)}$ denotes A with one superprime and k-1 super dots, and $A^{'(k)}$ denotes A with k super primes.

We will prove (3.5) for k = 2. The same method is valid for all k. First

$$\iiint \zeta_2^2\, t_{ij\omega,\alpha\beta}\, e_{ij\omega,\alpha\beta}\, dx_1\, dx_2\, dx_3$$

$$= \iiint \zeta_2^2\, t_{ij\omega,\alpha\beta}(e_{j\beta\omega,i\alpha} + e_{i\alpha\omega,\beta j} - e_{\alpha\beta\omega,ij} - c_{ij\alpha\beta\omega})\, dx_1 dx_2 dx_3$$

$$= \iiint \zeta_2^2\, t_{ij\omega,\alpha\beta}(2e_{i\alpha\omega,\beta j} - e_{\alpha\beta\omega,ij} - c_{ij\alpha\beta\omega})\, dx_1\, dx_2\, dx_3$$

using $(1.9)_3$ and $t_{ij\omega} = t_{ji\omega}$ where the region of integration is such that $|x_3| \leq h$ and $\zeta_2 \neq 0$.

Using the divergence theorem, $(1.9)_1$, and $(1.9)_2$, we obtain

$$\iiint \zeta_2^2\, t_{ij\omega,\alpha\beta}\, e_{ij\omega,\alpha\beta}\, dx_1\, dx_2\, dx_3$$

$$= \iiint [\zeta_2^2\, p_{ij\omega,j\alpha\beta}(2e_{i\alpha\omega,\beta} - e_{\alpha\beta\omega,i}) - (\zeta_2^2)_{,\gamma} t_{i\gamma\omega,\alpha\beta}(2e_{i\alpha\omega,\beta} - e_{\alpha\beta\omega,i})$$

$$- \zeta_2^2\, t_{ij\omega,\alpha\beta}\, c_{ij\alpha\beta\omega}]\, dx_1\, dx_2\, dx_3 \; .$$

Similarly

$$\iiint (\zeta_2^2)_{,\gamma}\, t_{i\gamma\omega,\alpha\beta}\, e_{\alpha\beta\omega,i}\, dx_1\, dx_2\, dx_3$$

$$= \iiint [(\zeta_2^2)_{,\gamma}\, p_{\gamma i\omega,i\alpha\beta}\, e_{\alpha\beta\omega} - (\zeta_2^2)_{,\gamma\nu}\, t_{\nu\gamma\omega,\alpha\beta}\, e_{\alpha\beta\omega}]\, dx_1\, dx_2\, dx_3 .$$

Thus

$$\iiint \zeta_2^2 \, t_{ij\omega,\alpha\beta} \, e_{ij\omega,\beta} \, dx_1 \, dx_2 \, dx_3$$

$$= \iiint [\zeta_2^2 \, p_{ij\omega,j\alpha\beta}(2e_{i\alpha\omega,\beta} - e_{\alpha\beta\omega,i}) - 2(\zeta_2^2)_{,\gamma} \, t_{i\gamma\omega,\alpha\beta} \, e_{i\alpha\omega,\beta}$$

$$+ (\zeta_2^2)_{,\gamma} \, p_{\gamma i\omega,i\alpha\beta} e_{\alpha\beta\omega} - (\zeta_2^2)_{,\gamma\nu} t_{\nu\gamma\omega,\alpha\beta} e_{\alpha\beta\omega} - \zeta_2^2 t_{ij\omega,\alpha\beta} c_{ij\alpha\beta\omega}] dx_1 dx_2 dx_3$$

$$= \iiint [-\zeta_2^2 p_{ij\omega,j\alpha}(2e_{i\alpha\omega,\beta\beta} - e_{\alpha\beta\omega,i\beta}) - (\zeta_2^2)_{,\beta} p_{ij\omega,j\alpha}(2e_{i\alpha\omega,\beta} - e_{\alpha\beta\omega,i})$$

$$- 2(\zeta_2^2)_{,\gamma} t_{i\gamma\omega,\alpha\beta} e_{i\alpha\omega,\beta} - (\zeta_2^2)_{,\gamma} p_{\gamma i\omega,i\alpha} \, e_{\alpha\beta\omega,\beta} - (\zeta_2^2)_{,\gamma\beta} p_{\gamma i\omega,i\alpha} e_{\alpha\beta\omega}$$

$$- (\zeta_2^2)_{,\gamma\nu} t_{\nu\gamma\omega,\alpha\beta} e_{\alpha\beta\omega} - \zeta_2^2 t_{ij\omega,\alpha\beta} c_{ij\alpha\beta\omega}] \, dx_1 \, dx_2 \, dx_3$$

using the divergence theorem again. This yields

$$\iiint \zeta_2^2 \, t_{ij\omega,\alpha\beta} \, e_{ij\omega,\alpha\beta} \, dx_1 \, dx_2 \, dx_3$$

$$= \mathcal{O}[\frac{d^2}{h^2} \, ||\zeta_2 \dot{p}'_\omega||_2^2 + ||\dot{\zeta}_2 \dot{e}_\omega||_2^2 + \frac{1}{d^4} \, ||e_\omega||_2^2 + ||\zeta_2 c_\omega||_2^2]$$

$$+ \sigma[\frac{h^2}{d^2}(||\zeta_2 e''_\omega||_2^2 + ||\dot{\zeta}_2 e'_\omega||_2^2) + ||\zeta_2 \ddot{t}_\omega||_2^2]$$

where we have used (3.2) and h/d $\leq$ 1.

Using $(3.4)_5$ and $(3.4)_6$,

$$\iiint \zeta_2^2 \, t_{ij\omega,\alpha\beta} \, e_{ij\omega,\alpha\beta} \, dx_1 \, dx_2 \, dx_3$$

$$= \mathcal{O}(\frac{d^2}{h^2} \, ||\zeta_2 \dot{p}'_\omega||_2^2 + \frac{1}{d^4} \, ||e_\omega||_2^2 + ||\zeta_2 c_\omega||_2^2)$$

$$+ \sigma(||\zeta_2 \ddot{t}_\omega||_2^2 + ||\zeta_2 \ddot{e}_\omega||_2^2 + \frac{h^2}{d^2} \, ||\zeta_2 e''_\omega||_2^2) \, .$$

From $(1.9)_4$,

$$t_{ij\omega,\alpha\beta}e_{ij\omega,\alpha\beta} = \left(\frac{\lambda}{E}\, e_{kk\omega,\alpha\beta}\delta_{ij} + \frac{2\mu}{E}\, e_{ij\omega,\alpha\beta} + z_{ij\omega,\alpha\beta}\right)e_{ij\omega,\alpha\beta}$$

$$\geq \frac{2\mu}{E}\, |\ddot{e}_\omega|^2 + z_{ij\omega,\alpha\beta}\, e_{ij\omega,\alpha\beta},$$

so that

$$|\ddot{e}_\omega|^2 = \mathcal{O}(t_{ij\omega,\alpha\beta}\, e_{ij\omega,\alpha\beta} + |\ddot{z}_\omega|^2) + \sigma(|\ddot{e}_\omega|^2)$$

and

$$|\ddot{e}_\omega|^2 = \mathcal{O}(t_{ij\omega,\alpha\beta}\, e_{ij\omega,\alpha\beta} + |\ddot{z}_\omega|^2) .$$

From $(1.9)_4$ we also have $|\ddot{t}_\omega|^2 = \mathcal{O}(|\ddot{e}_\omega|^2 + |\ddot{z}_\omega|^2)$. Combining with the above, we have

$$|\ddot{t}_\omega|^2 = \mathcal{O}(t_{ij\omega,\alpha\beta}\, e_{ij\omega,\alpha\beta} + |\ddot{z}_\omega|^2)$$

so that

$$||\zeta_2\ddot{t}_\omega||_2^2 = \mathcal{O}\Big(\frac{d^2}{h^2}\, ||\zeta_2\dot{p}'_\omega||_2^2 + \frac{1}{d^4}\, ||e_\omega||_2^2 + ||\zeta_2 c_\omega||_2^2 + ||\zeta_2\ddot{z}_\omega||_2^2\Big)$$

$$+ \sigma\Big(||\zeta_2\ddot{t}_\omega||_2^2 + ||\zeta_2\ddot{e}_\omega||_2^2 + \frac{h^2}{d^2}\, ||\zeta_2 e''_\omega||_2^2\Big)$$

which gives (3.5) with $k = 2$.

From $(1.9)_5$,

$$e_\omega = t_\omega + r_\omega .$$

Using the above, (1.20), and the fact that $(1.20)_3$ and $(1.20)_4$ can be written as (see end of Section 2)

$$c_1 = [t_1(t_1+t_2)\ F(t_1,t_2)]''$$

$$c_2 = [(t_1^2+t_2^2)\ F(t_1,t_2)]''$$

we can rewrite (3.5) as

$$||\zeta_k \dot{t}_1^{(k)}||_k = \mathcal{O}\{\frac{1}{d^2}||\dot{t}_1^{(k-2)}||_k + \frac{1}{d^2}||[t_1(t_1+t_2)F(t_1,t_2)]'^{(k-2)}||_k$$

$$+ \frac{d}{h}||\zeta_k([(t_1+t_2)t_1'+t_1t_2']F(t_1,t_2))^{\cdot(k-1)}||_k$$

$$+ ||\zeta_k[t_1(t_1+t_2)F(t_1,t_2)]'^{(k)}||_k\} + o(\frac{h}{d}||\zeta_k t_1'^{(k)}||_k) \ ;$$

(3.6)

$$||\zeta_k \dot{t}_2^{(k)}||_k = \mathcal{O}\{\frac{1}{d^2}||\dot{t}_2^{(k-2)}||_k + \frac{1}{d^2}||[(t_1^2+t_2^2)F(t_1,t_2)]'^{(k-2)}||_k$$

$$+ \frac{d}{h}||\zeta_k[(t_1+t_2)(t_1'+t_2')F(t_1,t_2)]^{\cdot(k-1)}||_k$$

$$+ ||\zeta_k[(t_1^2+t_2^2)F(t_1,t_2)]'^{(k)}||_k\} + o(\frac{h}{d}||\zeta_k t_2'^{(k)}||_k)$$

which is valid for k = 2,3,... .

Combining (3.6) with $(2.5)_1$ and $(2.5)_2$,

$$A_{k1} = \mathcal{O}\{\frac{1}{d^2}||\dot{t}_1^{(k-2)}||_k + \frac{1}{d^2}||[t_1(t_1+t_2)F(t_1,t_2)]'^{(k-2)}||_k$$

$$+ \frac{d}{h}||\zeta_k([(t_1+t_2)t_1'+t_1t_2']\ F(t_1,t_2))^{\cdot(k-1)}||_k$$

$$+ ||\zeta_k[t_1(t_1+t_2)F(t_1,t_2)]'^{(k)}||_k\} + o(\frac{h}{d}B_{k1})$$

(3.7)

$$A_{k2} = \mathcal{O}\{\frac{1}{d^2}||\dot{t}_2^{(k-2)}||_k + \frac{1}{d^2}||[(t_1^2+t_2^2)F(t_1,t_2)]'^{(k-2)}||_k$$

$$+ \frac{d}{h}||\zeta_k[(t_1+t_2)(t_1'+t_2')F(t_1,t_2)]^{\cdot(k-1)}||_k$$

$$+ ||\zeta_k[(t_1^2+t_2^2)F(t_1,t_2)]'^{(k)}||_k\} + o(\frac{h}{d}B_{k2}) \ ,$$

which is valid for k = 2,3,... .

From $(3.4)_{10}$,

$$||\zeta_k\{\dot{t}^{(k-1)}_{\alpha\beta\omega,3}\}||_k = \mathcal{O}\,[\frac{1}{hd}\,||\{\dot{t}^{(k-2)}_{\alpha\beta\omega}\}||_k + \frac{d}{h}\,||\zeta_k\{\dot{t}^{(k)}_{\alpha\beta\omega}\}||_k$$

$$+ \frac{h}{d}\,||\zeta_k\{\dot{t}^{(k-2)}_{\alpha\beta\omega,33}\}||_k]$$

$$= \mathcal{O}\,[\frac{1}{hd}\,||\dot{t}^{(k-2)}_{\omega}||_k + \frac{d}{h}\,A_{k\omega}]$$

where $\{\dot{t}^{(k-2)}_{\alpha\beta\omega,33}\}$ denotes the set of all functions of the type $\partial^k t_{\gamma\nu\omega}/\partial x_1^{k_1}\partial x_2^{k_2}\partial x_3^2$ $(\omega = 1,2)$, etc. Combining this with $(2.5)_3$ and $(2.5)_4$

$$B_{k1} = \mathcal{O}\,\{\frac{1}{hd}||\dot{t}^{(k-2)}_1||_k + \frac{d}{h}\,A_{k1} + ||\zeta_k\,[t_1(t_1+t_2)F(t_1,t_2)]'^{(k)}||_k\}$$

(3.8)

$$B_{k2} = \mathcal{O}\,\{\frac{1}{hd}||\dot{t}^{(k-2)}_2||_k + \frac{d}{h}\,A_{k2} + ||\zeta_k\,[(t_1^2+t_2^2)F(t_1,t_2)]'^{(k)}||_k\}$$

which is valid for $k = 2,3,\ldots$.

Next we show that

(3.9) $A_{\ell\omega} = \mathcal{O}(\frac{\varepsilon_\omega\sqrt{h}}{d^{\ell-1}})$, $B_{\ell\omega} = \mathcal{O}(\frac{\varepsilon_\omega}{d^{\ell-2}\sqrt{h}})$, $(\ell=1,2\ldots;\ \omega=1,2)$.

From $(1.11)_3$, $(1.11)_4$, and (0.9),

(3.10) $$|t_\omega| = \mathcal{O}\,(\varepsilon_\omega) \qquad (\omega=1,2).$$

From (3.4), (3.9), and (0.9),

$$||\zeta_2\dot{t}_1\dot{t}_1||_2 = \mathcal{O}\,(\varepsilon_1||\zeta_2\ddot{t}_1||_2 + \frac{\varepsilon_1^2\sqrt{h}}{d}) = \mathcal{O}\,(\varepsilon_1 A_{21} + \frac{\varepsilon_1^2\sqrt{h}}{d})$$

$$||\zeta_2 t_1' t_1'||_2 = \mathcal{O}\,(\varepsilon_1||\zeta_2 t_1''||_2 + \frac{\varepsilon_1^2 d}{h\sqrt{h}}) = \mathcal{O}\,(\varepsilon_1 B_{21} + \frac{\varepsilon_1^2 d}{h\sqrt{h}})$$

(3.11)

$$||\zeta_2\dot{t}_2\dot{t}_2||_2 = \mathcal{O}\,(\varepsilon_2||\zeta_2\ddot{t}_2||_2 + \frac{\varepsilon_2^2\sqrt{h}}{d}) = \mathcal{O}\,(\varepsilon_2 A_{22} + \frac{h^2\sqrt{h}\varepsilon_2}{d^3})$$

$$||\zeta_2 t_2' t_2'||_2 = \mathcal{O}\,(\varepsilon_2||\zeta_2 t_2''||_2 + \frac{\varepsilon_2^2 d}{h\sqrt{h}}) = \mathcal{O}\,(\varepsilon_2 B_{22} + \frac{\sqrt{h}\ \varepsilon_2}{d})$$

Next we look at the expressions appearing in (3.7) and (3.8). We list

$$\begin{aligned}
&([(t_1+t_2)t_1'+t_1t_2']F(t_1,t_2))^{\cdot} \\
&\quad = [(t_1+t_2)t_1''+t_1t_2''+(\dot{t}_1+\dot{t}_2)t_1'+\dot{t}_1t_2'+ t_1(t_2')^2]\ F(t_1,t_2); \\
&[t_1(t_1+t_2)F(t_1,t_2)]'' \\
&\quad = [(t_1+t_2)t_1''+t_1t_2''+(t_1'+t_2')t_1'+t_1(t_2')^2]\ F(t_1,t_2); \\
&[(t_1+t_2)(t_1'+t_2')F(t_1,t_2)]^{\cdot} \\
&\quad = [(t_1+t_2)(t_1''+t_2'')+(t_1'+t_2')(\dot{t}_1+\dot{t}_2)]F(t_1,t_2); \\
&[(t_1^2+t_2^2)F(t_1,t_2)]'' \\
&\quad = [(t_1+t_2)(t_1''+t_2'') + (t_1'+t_2')^2]\ F(t_1,t_2).
\end{aligned} \tag{3.12}$$

From (3.12), (3.10), and (0.9),

$$\begin{aligned}
&|([(t_1+t_2)t_1'+t_1t_2']F(t_1,t_2))^{\cdot}| \\
&\quad = \mathcal{O}[\theta^2|t_1''|+\varepsilon_1|t_2''| + \frac{d}{h}|\dot{t}_1|^2 + \frac{h}{d}\ |t_1'|^2 \\
&\qquad + \frac{\varepsilon_1}{\varepsilon_2}\,\frac{d}{h}|\dot{t}_2|^2+ \frac{\varepsilon_2}{\varepsilon_1}\,\frac{h}{d}|t_1'|^2+ \frac{\varepsilon_2}{\varepsilon_1}\,\frac{d}{h}|\dot{t}_1|^2+ \frac{\varepsilon_1}{\varepsilon_2}\,\frac{h}{d}|t_2'|^2+\varepsilon_1|t_2'|^2] \\
&|[t_1(t_1+t_2)F(t_1,t_2)]''| \\
&\quad = \mathcal{O}[\theta^2|t_1''|+\varepsilon_1|t_2''|+|t_1'|^2+ \frac{\varepsilon_1}{\varepsilon_2}\ |t_2'|^2+ \frac{\varepsilon_2}{\varepsilon_1}|t_1'|^2+ \varepsilon_1|t_2'|^2] \\
&|[(t_1+t_2)(t_1'+t_2')F(t_1,t_2)]^{\cdot}| \\
&\quad = \mathcal{O}[\theta^2|t_2''|+(\varepsilon_1+\varepsilon_2)|t_1''|+ \frac{h}{d}|t_1'|^2+ \frac{d}{h}|\dot{t}_1|^2+ \frac{h}{d}|t_2'|^2+ \frac{d}{h}|\dot{t}_2|^2] \\
&|[(t_1^2+t_2^2)F(t_1,t_2)]''| \\
&\quad = \mathcal{O}[\theta^2|t_2''|+(\varepsilon_1+\varepsilon_2)|t_1''| + |t_1'|^2+ |t_2'|^2].
\end{aligned}$$

From the above, (3.11) and (0.9),

$$\begin{aligned}
&\|\zeta_2([(t_1+t_2)t_1'+t_1t_2']\,F(t_1,t_2))^{\cdot}\|_2 \\
&\quad = \mathcal{O}\Big[\theta^2B_{21} + \varepsilon_1B_{22} + \theta\theta_0\,A_{21} + \frac{h\sqrt{h}\,\varepsilon_1}{d^2} + \varepsilon_1\,\frac{d}{h}\,A_{22}\Big] \\
&\|\zeta_2[t_1(t_1+t_2)F(t_1,t_2)]''\|_2 \\
&\quad = \mathcal{O}\Big[\theta^2B_{21} + \varepsilon_1B_{22} + \frac{\sqrt{h}\,\varepsilon_1}{d}\Big] \\
&\|\zeta_2[(t_1+t_2)(t_1'+t_2')F(t_1,t_2)]^{\cdot}\|_2 \\
&\quad = \mathcal{O}\Big[\theta^2B_{22} + (\varepsilon_1+\varepsilon_2)B_{21} + \frac{d}{h}\,\varepsilon_1A_{21} + \theta_0\theta\,A_{22} + \frac{h\sqrt{h}}{d^2}\,\varepsilon_2\Big] \\
&\|\zeta_2[(t_1^2+t_2^2)F(t_1,t_2)]''\|_2 \\
&\quad = \mathcal{O}\Big[\theta^2B_{22} + (\varepsilon_1+\varepsilon_2)B_{21} + \frac{\sqrt{h}\varepsilon_2}{d}\Big]
\end{aligned} \tag{3.13}$$

Substituting (3.13) into (3.7) and (3.8)

$$\begin{aligned}
A_{21} &= \mathcal{O}\Big[\frac{\sqrt{h}\,\varepsilon_1}{d} + \theta_0\theta B_{21} + \frac{d}{h}\,\varepsilon_1B_{22} + \theta_0^2A_{21} + \varepsilon_1\,\frac{d^2}{h^2}\,A_{22}\Big] + o\Big(\frac{h}{d}B_{21}\Big) \\
A_{22} &= \mathcal{O}\Big[\frac{\sqrt{h}\,\varepsilon_2}{d} + \theta_0\theta B_{22} + \frac{d}{h}(\varepsilon_1+\varepsilon_2)B_{21} + \frac{d^2}{h^2}\,\varepsilon_1A_{21} + \theta_0^2A_{22}\Big] + o\Big(\frac{h}{d}B_{22}\Big) \\
B_{21} &= \mathcal{O}\Big(\frac{\varepsilon_1}{\sqrt{h}} + \frac{d}{h}\,A_{21} + \theta_0^2B_{21} + \varepsilon_1B_{22}\Big) \\
B_{22} &= \mathcal{O}\Big[\frac{\varepsilon_2}{\sqrt{h}} + \frac{d}{h}\,A_{22} + \theta_0^2B_{22} + (\varepsilon_1+\varepsilon_2)B_{21}\Big]
\end{aligned} \tag{3.14}$$

$(3.14)_4$ has the form

$$B_{22} \le \kappa\Big[\frac{\varepsilon_2}{\sqrt{h}} + \frac{d}{h}\,A_{22} + \theta_0^2B_{22} + (\varepsilon_1+\varepsilon_2)B_{21}\Big]$$

where $\kappa \ge 0$ and can be chosen to be independent of θ_0. Thus if we pick θ_0 so that $\kappa\theta_0^2 \le 1/2$ we have

$$B_{22} \leq 2\kappa\left[\frac{\varepsilon_2}{\sqrt{h}} + \frac{d}{h} A_{22} + (\varepsilon_1+\varepsilon_2)B_{21}\right] .$$

Substituting this into (3.14) and using (0.9) gives

$$\begin{aligned}
A_{21} &= \mathcal{O}\left[\frac{\sqrt{h}\,\varepsilon_1}{d} + \theta_0\theta B_{21} + \theta_0^2 A_{21} + \varepsilon_1 \frac{d^2}{h^2} A_{22}\right] + o\left(\frac{h}{d} B_{21}\right) \\
A_{22} &= \mathcal{O}\left[\frac{\sqrt{h}\,\varepsilon_2}{d} + \frac{d}{h}(\varepsilon_1+\varepsilon_2)B_{21} + \frac{d^2}{h^2}\varepsilon_1 A_{21} + \theta_0^2 A_{22}\right] + o(A_{22}) \\
B_{21} &= \mathcal{O}\left[\frac{\varepsilon_1}{\sqrt{h}} + \frac{d}{h} A_{21} + \theta_0^2 B_{21} + \varepsilon_1 \frac{d}{h} A_{22}\right] \\
B_{22} &= \mathcal{O}\left[\frac{\varepsilon_2}{\sqrt{h}} + \frac{d}{h} A_{22} + (\varepsilon_1+\varepsilon_2)B_{21}\right] .
\end{aligned} \tag{3.15}$$

$(3.15)_2$ has the form

$$A_{22} \leq \kappa(k)\left[\frac{\sqrt{h}\,\varepsilon_2}{d} + \frac{d}{h}(\varepsilon_1+\varepsilon_2)B_{21} + \frac{d^2}{h^2}\varepsilon_1 A_{21} + \theta_0^2 A_{22}\right] + kA_{22}$$

where $\kappa(k) \geq 0$ and can be chosen to be independent of θ_0. Therefore

$$A_{22} \leq 2\kappa(\tfrac{1}{2})\left[\frac{\sqrt{h}\,\varepsilon_2}{d} + \frac{d}{h}(\varepsilon_1+\varepsilon_2)B_{21} + \frac{d^2}{h^2}\varepsilon_1 A_{21} + \theta_0^2 A_{22}\right] .$$

For $2\kappa(1/2)\theta_0^2 \leq 1/2$, this becomes

$$A_{22} \leq 4\kappa(\tfrac{1}{2})\left[\frac{\sqrt{h}\,\varepsilon_2}{d} + \frac{d}{h}(\varepsilon_1+\varepsilon_2)B_{21} + \frac{d^2}{h^2}\varepsilon_1 A_{21}\right] .$$

Substituting into (3.15)

$$\begin{aligned}
A_{21} &= \mathcal{O}\left[\frac{\sqrt{h}\,\varepsilon_1}{d} + \theta_0\theta B_{21} + \theta_0^2 A_{21}\right] + o\left(\frac{h}{d} B_{21}\right) \\
A_{22} &= \mathcal{O}\left[\frac{\sqrt{h}\,\varepsilon_2}{d} + \frac{d}{h}(\varepsilon_1+\varepsilon_2)B_{21} + \frac{d^2}{h^2}\varepsilon_1 A_{21}\right] \\
B_{21} &= \mathcal{O}\left[\frac{\varepsilon_1}{\sqrt{h}} + \frac{d}{h} A_{21} + \theta_0^2 B_{21}\right] \\
B_{22} &= \mathcal{O}\left[\frac{\varepsilon_2}{\sqrt{h}} + \frac{d^2}{h^2}(\varepsilon_1+\varepsilon_2)B_{21} + \frac{d^3}{h^3}\varepsilon_1 A_{21}\right] .
\end{aligned} \tag{3.16}$$

$(3.16)_3$ gives $B_{21} = \mathcal{O}(\frac{\varepsilon_1}{\sqrt{h}} + \frac{d}{h} A_{21})$. Substituting into (3.16) yields

$$(3.17) \qquad \begin{aligned} A_{21} &= \mathcal{O}\,[\frac{\sqrt{h}\ \varepsilon_1}{d} + \theta_0^2 A_{21}] + o(A_{21}) \\ A_{22} &= \mathcal{O}\,[\frac{\sqrt{h}\ \varepsilon_2}{d} + \frac{d^2}{h^2}\,(\varepsilon_1+\varepsilon_2)A_{21}] \\ B_{21} &= \mathcal{O}\,[\frac{\varepsilon_1}{\sqrt{h}} + \frac{d}{h}\,A_{21}] \\ B_{22} &= \mathcal{O}\,[\frac{\varepsilon_2}{\sqrt{h}} + \frac{d^3}{h^3}\,(\varepsilon_1+\varepsilon_2)A_{21}] \end{aligned}$$

$(3.17)_1$ gives $A_{21} = \mathcal{O}(\sqrt{h}\varepsilon_1/d)$. Substituting into (3.17) gives (3.9) for $\ell = 2$.

Next we prove (3.9) for $\ell = 1$. From $(3.4)_1$ and (3.9) for $\ell=2$

$$||\zeta_1\dot{t}_\omega||_1 = \mathcal{O}(\frac{1}{d}\ ||t_\omega||_1) + o(d||\zeta_2\ddot{t}_\omega||_2) \qquad \text{[recall that } \zeta_1 = \zeta_2]$$

$$= \mathcal{O}(\varepsilon_\omega\sqrt{h} + d\ A_{2\omega}) \ = \mathcal{O}\ (\sqrt{h}\ \varepsilon_\omega)\ .$$

From $(3.4)_3$ and (3.9) for $\ell = 2$

$$||\zeta_1 t'_\omega||_1 = \mathcal{O}(\frac{1}{h}\ ||t_\omega||_1) + o(d||\zeta_2\ddot{t}_\omega||_2 + h||\zeta_2 t''_\omega||_2)$$

$$= \mathcal{O}\ (\frac{d\varepsilon_\omega}{\sqrt{h}} + dA_{2\omega} + hB_{2\omega}) \ = \mathcal{O}\ (\frac{d\varepsilon_\omega}{\sqrt{h}})\ .$$

Also

$$[t_1(t_1+t_2)F(t_1,t_2)]' = [(t_1+t_2)t_1' + t_1t_2']F(t_1,t_2)$$

$$||\zeta_1[t_1(t_1+t_2)F(t_1,t_2)]'||_1 = \mathcal{O}\ [(\varepsilon_1+\varepsilon_2)||\zeta_1 t_1'||_1 + \varepsilon_1||\zeta_1 t_2'||_1]$$

$$= \mathcal{O}\ [(\varepsilon_1+\varepsilon_2)\ \frac{d\varepsilon_1}{\sqrt{h}}\ ] = \mathcal{O}\ (\frac{h\sqrt{h}}{d}\ \varepsilon_1)$$

$$[(t_1^2+t_2^2)F(t_1,t_2)]' = (t_1+t_2)(t_1'+t_2')F(t_1,t_2)$$

$$||\zeta_1[(t_1^2+t_2^2)F(t_1,t_2)]'|| = \mathcal{O}[(\varepsilon_1+\varepsilon_2)(||\zeta_1 t_1'||_1+||\zeta_1 t_2'||_1)]$$

$$= \mathcal{O}[\frac{d}{\sqrt{h}}(\varepsilon_1+\varepsilon_2)^2] = \mathcal{O}(\frac{h\sqrt{h}}{d}\varepsilon_2) .$$

Using these results in (2.5) with k = 1, we obtain (3.9) with $\ell = 1$.

Next we prove (3.9) with $\ell = 3$. From (3.11) and (3.9) ($\ell = 2$) we obtain

$$||\zeta_2\dot{t}_\omega\dot{t}_\omega|| = \mathcal{O}(\frac{h^2\sqrt{h}\,\varepsilon_\omega}{d^3})$$

(3.18)

$$||\zeta_2 t_\omega' t_\omega'||_2 = \mathcal{O}(\frac{\sqrt{h}\,\varepsilon_\omega}{d})$$

For the present argument we let

$$m_\omega = \max \zeta_3|\dot{t}_\omega|$$

(3.19)

$$M_\omega = \max \zeta_3|t_\omega'|$$

From (3.12)

$$([(t_1+t_2)t_1'+t_1t_2']F(t_1,t_2))^{\cdot\cdot}$$

$$= [(t_1+t_2)t_1''' + t_1t_2''' + (\dot{t}_1+\dot{t}_2)t_1'' + \dot{t}_1t_2'' + (\ddot{t}_1+\ddot{t}_2)t_1'$$

$$+\ddot{t}_1t_2' + (t_1'+t_2')^2t_1' + t_1t_2'(t_2''+t_2'^2)]F(t_1,t_2)$$

$$[t_1(t_1+t_2)F(t_1,t_2)]'''$$

(3.20)

$$= [(t_1+t_2)t_1''' + t_1t_2''' + (t_1'+t_2')t_1'' + t_1't_2''$$

$$+(t_1'+t_2')^2t_1' + t_1t_2'(t_2''+t_2'^2)]F(t_1,t_2)$$

$$[(t_1+t_2)(t_1'+t_2')F(t_1,t_2)]^{\cdot\cdot}$$

$$= [(t_1+t_2)(t_1'''+t_2''') + (\dot{t}_1+\dot{t}_2)(t_1''+t_2'') + (t_1'+t_2')(\ddot{t}_1+\ddot{t}_2)$$

$$+ (t_1'+t_2')^3] F(t_1,t_2)$$

$$[(t_1^2+t_2^2)F(t_1,t_2)]'''$$

$$= [(t_1+t_2)(t_1'''+t_2''')+(t_1'+t_2')(t_1''+t_2'')+(t_1'+t_2')^3]F(t_1,t_2) \ .$$

From (3.20) and the fact that $\zeta_3 = \mathcal{O}(\zeta_3\zeta_2)$ we obtain

$$\zeta_3|([(t_1+t_2)t_1'+t_1t_2']F(t_1,t_2))^{\cdot\cdot}|$$

$$= \mathcal{O}\ [\theta^2\zeta_3|t_1'''|+\varepsilon_1\zeta_3|t_2'''|+(m_1+m_2)\zeta_2|t_1''|+m_1\zeta_2|t_2''|+M_1\zeta_2(|\ddot{t}_1|+|\ddot{t}_2|)$$

$$+ M_2\zeta_2|\ddot{t}_1|+M_1\zeta_2(|t_1'|^2+|t_2'|^2)+\varepsilon_1M_2\zeta_2(|t_2''|+|t_2'|^2)]$$

$$\zeta_3|[t_1(t_1+t_2)F(t_1,t_2)]'''|$$

$$= \mathcal{O}\ [\theta^2\zeta_3|t_1'''|+\varepsilon_1\zeta_3|t_2'''|+(M_1+M_2)\zeta_2|t_1''|+M_1\zeta_2|t_2''|$$

$$+ M_1\zeta_2(|t_1'|^2+|t_2'|^2) + \varepsilon_1M_2\zeta_2(|t_2''|+|t_2'|^2)]$$

$$\zeta_3|[(t_1+t_2)(t_1'+t_2')F(t_1,t_2)]^{\cdot\cdot}|$$

$$= \mathcal{O}\ [(\varepsilon_1+\varepsilon_2)\zeta_3|t_1'''|+ \theta^2\zeta_3|t_2'''|+(m_1+m_2)\zeta_2(|t_1''|+|t_2''|)$$

$$+ (M_1+M_2)\zeta_2(|\ddot{t}_1|+|\ddot{t}_2|)+(M_1+M_2)\zeta_2(|t_1'|^2+|t_2'|^2)]$$

$$\zeta_3|[(t_1^2+t_2^2)F(t_1,t_2)]'''|$$

$$= \mathcal{O}\ [(\varepsilon_1+\varepsilon_2)\zeta_3|t_1'''|+ \theta^2\zeta_3|t_2'''|+(M_1+M_2)\zeta_2(|t_1''|+|t_2''|)$$

$$+ (M_1+M_2)\zeta_2(|t_1'|^2+|t_2'|^2)] \ .$$

Using the above, (3.9) for $\ell = 2$, and (3.18)

$$||\zeta_3([(t_1+t_2)t_1'+t_1t_2']F(t_1,t_2))^{\cdot\cdot}||_3$$

$$= \mathcal{O}\ [\theta^2B_{31}+\varepsilon_1B_{32} + (m_1+m_2)\ \frac{\varepsilon_1}{\sqrt{\hbar}} + m_1\ \frac{\varepsilon_2}{\sqrt{\hbar}} +M_1\ \frac{\sqrt{\hbar}}{d}(\varepsilon_1+\varepsilon_2)+M_2\ \frac{\sqrt{\hbar}}{d}\ \varepsilon_1]$$

$$||\zeta_3\ [t_1(t_1+t_2)F(t_1,t_2)]'''||_3$$

$$(3.21) = \mathcal{O}\ [\theta^2B_{31}+\varepsilon_1B_{32}+(M_1+M_2)\ \frac{\varepsilon_1}{\sqrt{\hbar}} + M_1\ \frac{\varepsilon_2}{\sqrt{\hbar}}\ ]$$

$$||\zeta_3[(t_1+t_2)(t_1'+t_2')F(t_1,t_2)]^{\cdot\cdot}||_3$$

$$= \mathcal{O}\,[(\varepsilon_1+\varepsilon_2)B_{31} + \theta^2B_{32}+(m_1+m_2)\,\frac{1}{\sqrt{h}}\,(\varepsilon_1+\varepsilon_2)+(M_1+M_2)\frac{\sqrt{h}}{d}(\varepsilon_1+\varepsilon_2)]$$

$$||\zeta_3[(t_1^2+t_2^2)F(t_1,t_2)]'''||_3$$

$$= \mathcal{O}\,[(\varepsilon_1+\varepsilon_2)B_{31} + \theta^2B_{32} + (M_1+M_2)\,\frac{1}{\sqrt{h}}\,(\varepsilon_1+\varepsilon_2)]\ .$$

Also

$$||\dot{t}_1||_3 = \mathcal{O}(||\zeta_1\dot{t}_1||_1) = \mathcal{O}(A_{11}) = \mathcal{O}(\sqrt{h}\,\varepsilon_1)$$

$$||[t_1(t_1+t_2)F(t_1,t_2)]'||_3 = ||[(t_1+t_2)t_1'+t_1t_2']F(t_1,t_2)||_3$$

$$= \mathcal{O}(||\theta^2|t_1'|+\varepsilon_1|t_2'|\ ||_3) = \mathcal{O}(\theta^2||\zeta_1t_1'||_1+\varepsilon_1||\zeta_1t_2'||_1)$$

$$= \mathcal{O}(\theta^2B_{11}+\varepsilon_1B_{12}) \quad = \mathcal{O}(\frac{h\sqrt{h}}{d}\,\varepsilon_1)$$

$$||\dot{t}_2||_3 = \mathcal{O}(A_{12}) = \mathcal{O}(\varepsilon_2\sqrt{h})$$

$$||[(t_1^2+t_2^2)F(t_1,t_2)]'||_3 = \mathcal{O}||(t_1+t_2)(t_1'+t_2')F(t_1,t_2)||_3$$

$$= \mathcal{O}\,(\varepsilon_1+\varepsilon_2)(B_{11}+B_{12}) = \mathcal{O}\,[\frac{d}{\sqrt{h}}\,(\varepsilon_1+\varepsilon_2)^2] = \mathcal{O}\,(\frac{h\sqrt{h}}{d}\,\varepsilon_2)\ .$$

From the above, (3.21), (3.7), and (3.8)

$$\begin{aligned}
A_{31} &= \mathcal{O}\,[\frac{\sqrt{h}\,\varepsilon_1}{d^2} + \theta_0\theta B_{31} + (m_1+m_2)\,\frac{d\varepsilon_1}{h\sqrt{h}} + m_1\,\frac{d\varepsilon_2}{h\sqrt{h}} + M_1\,\frac{1}{\sqrt{h}}(\varepsilon_1+\varepsilon_2) \\
&\quad + M_2\,\frac{\varepsilon_1}{\sqrt{h}} + \frac{d}{h}\,\varepsilon_1B_{32}] + o(\frac{h}{d}\,B_{31}) \\
A_{32} &= \mathcal{O}\,\{\frac{\sqrt{h}\,\varepsilon_2}{d^2} + \theta_0\theta B_{32} + (\varepsilon_1+\varepsilon_2)\,\frac{d}{h}[B_{31}+(m_1+m_2)\frac{1}{\sqrt{h}} + (M_1+M_2)\frac{\sqrt{h}}{d}]\} \\
&\quad + o(\frac{h}{d}\,B_{32}) \\
B_{31} &= \mathcal{O}\,[\frac{\varepsilon_1}{d\sqrt{h}} + \frac{d}{h}\,A_{31} + \theta_0^2B_{31} + \varepsilon_1B_{32} + (M_1+M_2)\,\frac{\varepsilon_1}{\sqrt{h}} + M_1\,\frac{\varepsilon_2}{\sqrt{h}}] \\
B_{32} &= \mathcal{O}\,\{\frac{\varepsilon_2}{d\sqrt{h}} + \frac{d}{h}\,A_{32} + \theta_0^2B_{32} + (\varepsilon_1+\varepsilon_2)\,[B_{31}+\frac{1}{\sqrt{h}}\,(M_1+M_2)]\}
\end{aligned} \tag{3.22}$$

From $(3.22)_4$

$$B_{32} = \mathcal{O}\,\{\frac{\varepsilon_2}{d\sqrt{h}} + \frac{d}{h}\,A_{32} + (\varepsilon_1+\varepsilon_2)\,[B_{31} + \frac{1}{\sqrt{h}}\,(M_1+M_2)]\}.$$

for θ_0 sufficiently small. Substituting into (3.22) gives

$$A_{31} = \mathcal{C}\ [\frac{\sqrt{h}\ \varepsilon_1}{d^2} + \theta_0\theta B_{31} + \frac{d^2}{h^2}\varepsilon_1 A_{32} + (m_1+m_2)\frac{d\varepsilon_1}{h\sqrt{h}} + m_1\frac{d\varepsilon_2}{h\sqrt{h}}$$

$$+ M_1\frac{1}{\sqrt{h}}(\varepsilon_1+\varepsilon_2) + M_2\frac{\varepsilon_1}{\sqrt{h}}] + o(\frac{h}{d}B_{31})$$

$$\text{(3.23)}\quad A_{32} = \mathcal{C}\{\frac{\sqrt{h}\varepsilon_2}{d^2} + \theta_0^2 A_{32} + (\varepsilon_1+\varepsilon_2)\frac{d}{h}[B_{31} + (m_1+m_2)\frac{1}{\sqrt{h}} + (M_1+M_2)\frac{\sqrt{h}}{d}]\} + o(A_{32})$$

$$B_{31} = \mathcal{C}\ [\frac{\varepsilon_1}{d\sqrt{h}} + \frac{d}{h}A_{31} + \theta_0^2 B_{31} + \varepsilon_1\frac{d}{h}A_{32} + (M_1+M_2)\frac{\varepsilon_1}{\sqrt{h}} + M_1\frac{\varepsilon_2}{\sqrt{h}}]$$

$$B_{32} = \mathcal{C}\ \{\frac{\varepsilon_2}{d\sqrt{h}} + \frac{d}{h}A_{32} + (\varepsilon_1+\varepsilon_2)[B_{31} + \frac{1}{\sqrt{h}}(M_1+M_2)]\}$$

From $(3.23)_2$

$$A_{32} = \mathcal{C}\{\frac{\sqrt{h}\varepsilon_2}{d^2} + (\varepsilon_1+\varepsilon_2)\frac{d}{h}[B_{31} + (m_1+m_2)\frac{1}{\sqrt{h}} + (M_1+M_2)\frac{\sqrt{h}}{d}]\}\ \text{for}\ \theta_0$$

sufficiently small. Substituting into (3.23)

$$A_{31} = \mathcal{C}\ [\frac{\sqrt{h}\ \varepsilon_1}{d^2} + \theta_0\theta B_{31} + (m_1+m_2)\frac{d\varepsilon_1}{h\sqrt{h}} + m_1\frac{d\varepsilon_2}{h\sqrt{h}}$$

$$+ M_1\frac{1}{\sqrt{h}}(\varepsilon_1+\varepsilon_2) + M_2\frac{\varepsilon_1}{\sqrt{h}}] + o(\frac{h}{d}B_{31})$$

$$\text{(3.24)}\quad A_{32} = \mathcal{C}\{\frac{\sqrt{h}\ \varepsilon_2}{d^2} + (\varepsilon_1+\varepsilon_2)\frac{d}{h}[B_{31} + (m_1+m_2)\frac{1}{\sqrt{h}} + (M_1+M_2)\frac{\sqrt{h}}{d}]\}$$

$$B_{31} = \mathcal{C}\ [\frac{\varepsilon_1}{d\sqrt{h}} + \frac{d}{h}A_{31} + \theta_0^2 B_{31} + (M_1+M_2)\frac{\varepsilon_1}{\sqrt{h}} + M_1\frac{\varepsilon_2}{\sqrt{h}}]$$

$$B_{32} = \mathcal{C}\{\frac{\varepsilon_2}{d\sqrt{h}} + (\varepsilon_1+\varepsilon_2)\frac{d^2}{h^2}[B_{31} + (m_1+m_2)\frac{1}{\sqrt{h}} + (M_1+M_2)\frac{\sqrt{h}}{d}]\}$$

From $(3.24)_3$

$$B_{31} = \mathcal{C}\ [\frac{\varepsilon_1}{d\sqrt{h}} + \frac{d}{h}A_{31} + (M_1+M_2)\frac{\varepsilon_1}{\sqrt{h}} + M_1\frac{\varepsilon_2}{\sqrt{h}}]\ \text{for}\ \theta_0$$

sufficiently small. Substituting into (3.24),

$$A_{31} = \mathcal{O}\Big[\frac{\sqrt{h}\,\varepsilon_1}{d^2} + \theta_0^2 A_{31} + (m_1+m_2)\frac{d\varepsilon_1}{h\sqrt{h}} + m_1\,\frac{d\varepsilon_2}{h\sqrt{h}}$$

$$+ M_1\,\frac{1}{\sqrt{h}}\,(\varepsilon_1+\varepsilon_2) + M_2\,\frac{\varepsilon_1}{\sqrt{h}}\Big] + o(A_{31})$$

$$(3.25)\quad A_{32} = \mathcal{O}\Big\{\frac{\sqrt{h}\,\varepsilon_2}{d^2} + (\varepsilon_1+\varepsilon_2)\,\frac{d}{h}\Big[\frac{d}{h}\,A_{31}+(m_1+m_2)\frac{1}{\sqrt{h}}+(M_1+M_2)\frac{\sqrt{h}}{d}\Big]\Big\}$$

$$B_{31} = \mathcal{O}\Big[\frac{\varepsilon_1}{d\sqrt{h}} + \frac{d}{h}\,A_{31} + (M_1+M_2)\,\frac{\varepsilon_1}{\sqrt{h}} + M_1\,\frac{\varepsilon_2}{\sqrt{h}}\Big]$$

$$B_{32} = \mathcal{O}\Big\{\frac{\varepsilon_2}{d\sqrt{h}} + (\varepsilon_1+\varepsilon_2)\,\frac{d^2}{h^2}\Big[\frac{d}{h}\,A_{31}+(m_1+m_2)\frac{1}{\sqrt{h}}+(M_1+M_2)\,\frac{\sqrt{h}}{d}\Big]\Big\}.$$

From $(3.25)_1$

$$A_{31} = \mathcal{O}\Big[\frac{\sqrt{h}\,\varepsilon_1}{d^2} + (m_1+m_2)\,\frac{d\varepsilon_1}{h\sqrt{h}} + m_1\,\frac{d\varepsilon_2}{h\sqrt{h}} + M_1\,\frac{1}{\sqrt{h}}\,(\varepsilon_1+\varepsilon_2)+M_2\,\frac{\varepsilon_1}{\sqrt{h}}\Big],$$

for θ_0 sufficiently small. Substituting into (3.25),

$$\begin{aligned}
A_{31} &= \mathcal{O}\Big[\frac{\sqrt{h}\,\varepsilon_1}{d^2} +(m_1+m_2)\,\frac{d\varepsilon_1}{h\sqrt{h}} + m_1\,\frac{d\varepsilon_2}{h\sqrt{h}} +M_1\,\frac{1}{\sqrt{h}}\,(\varepsilon_1+\varepsilon_2)+M_2\,\frac{\varepsilon_1}{\sqrt{h}}\Big]\\
A_{32} &= \mathcal{O}\Big\{\frac{\sqrt{h}\,\varepsilon_2}{d^2} +(\varepsilon_1+\varepsilon_2)\,\frac{d}{h}\Big[(m_1+m_2)\,\frac{1}{\sqrt{h}} +(M_1+M_2)\,\frac{\sqrt{h}}{d}\Big]\Big\}\\
B_{31} &= \mathcal{O}\Big[\frac{\varepsilon_1}{d\sqrt{h}} +(m_1+m_2)\,\frac{d^2\varepsilon_1}{h^2\sqrt{h}} +m_1\,\frac{d^2\varepsilon_2}{h^2\sqrt{h}} + M_1\,\frac{d}{h\sqrt{h}}\,(\varepsilon_1+\varepsilon_2)+M_2\frac{d}{h\sqrt{h}}\varepsilon_1\Big]\\
B_{32} &= \mathcal{O}\Big\{\frac{\varepsilon_2}{d\sqrt{h}} +(\varepsilon_1+\varepsilon_2)\,\frac{d^2}{h^2}\,\Big[(m_1+m_2)\,\frac{1}{\sqrt{h}} +(M_1+M_2)\,\frac{\sqrt{h}}{d}\Big]\Big\}
\end{aligned}\qquad (3.26)$$

From $(3.4)_{11}$

$$\begin{aligned}
\frac{\sqrt{h}}{d}\,\zeta_3|\dot{t}_\omega| &= \mathcal{O}\,\Big(A_{3\omega} + \frac{1}{d^2}\,||\dot{t}_\omega||_3 + \frac{h^2}{d^2}\,B_{3\omega}\Big)\\
&= \mathcal{O}\,\Big(A_{3\omega}+ \frac{1}{d^2}||\zeta_1\dot{t}_\omega||_1+ \frac{h^2}{d^2}\,B_{3\omega}\Big) =\mathcal{O}\,\Big(A_{3\omega}+ \frac{1}{d^2}\,A_{1\omega}+ \frac{h^2}{d^2}\,B_{3\omega}\Big)\\
&= \mathcal{O}\,\Big(\frac{\sqrt{h}\,\varepsilon_\omega}{d^2} + A_{3\omega} + \frac{h^2}{d^2}\,B_{3\omega}\Big)
\end{aligned}$$

so that

$$m_\omega = \mathcal{O}\,\Big(\frac{\varepsilon_\omega}{d} + \frac{d}{\sqrt{h}}\,A_{3\omega}+ \frac{h\sqrt{h}}{d}\,B_{3\omega}\Big)\ .$$

Similarly from (3.4)$_{11}$

$$\begin{aligned}
\frac{\sqrt{h}}{d}\zeta_3|t_\omega'| &= \mathcal{O}[||\zeta_3 t_\omega'''||_3 + \frac{1}{d^2}||t_\omega'||_3] \\
&= \mathcal{O}(B_{3\omega} + \frac{1}{d^2}||\zeta_1 t_\omega'||_1) \\
&= \mathcal{O}(B_{3\omega} + \frac{1}{d^2}B_{1\omega}) \\
&= \mathcal{O}(\frac{\varepsilon_\omega}{d\sqrt{h}} + B_{3\omega})
\end{aligned}$$

so that

$$M_\omega = \mathcal{O}(\frac{\varepsilon_\omega}{h} + \frac{d}{\sqrt{h}}B_{3\omega}) \ .$$

Using (3.26) these become

$$\begin{aligned}
m_1 &= \mathcal{O}[\frac{\varepsilon_1}{d} + (m_1+m_2)\frac{d^2}{h^2}\varepsilon_1 + m_1\frac{d^2}{h^2}\varepsilon_2 + M_1\frac{d}{h}(\varepsilon_1+\varepsilon_2) + M_2\frac{d}{h}\varepsilon_1] \\
m_2 &= \mathcal{O}\{\frac{\varepsilon_2}{d} + (\varepsilon_1+\varepsilon_2)\frac{d^2}{h^2}[m_1+m_2+\frac{h}{d}(M_1+M_2)]\} \\
M_1 &= \mathcal{O}[\frac{\varepsilon_1}{h} + (m_1+m_2)\frac{d^3}{h^3}\varepsilon_1 + m_1\frac{d^3}{h^3}\varepsilon_2 + M_1\frac{d^2}{h^2}(\varepsilon_1+\varepsilon_2) + M_2\frac{d^2}{h^2}\varepsilon_1] \\
M_2 &= \mathcal{O}\{\frac{\varepsilon_2}{h} + (\varepsilon_1+\varepsilon_2)\frac{d^3}{h^3}[m_1+m_2+\frac{h}{d}(M_1+M_2)]\}
\end{aligned} \tag{3.27}$$

We can treat (3.27) just as we treated (3.22). For θ_0 sufficiently small we can eliminate m_1 from the right-hand side of (3.27) to get

$$\begin{aligned}
m_1 &= \mathcal{O}[\frac{\varepsilon_1}{d} + m_2\frac{d^2}{h^2}\varepsilon_1 + M_1\frac{d}{h}(\varepsilon_1+\varepsilon_2) + M_2\frac{d}{h}\varepsilon_1] \\
m_2 &= \mathcal{O}\{\frac{\varepsilon_2}{d} + (\varepsilon_1+\varepsilon_2)\frac{d^2}{h^2}[m_2+\frac{h}{d}(M_1+M_2)]\} \\
M_1 &= \mathcal{O}[\frac{\varepsilon_1}{h} + m_2\frac{d^3}{h^3}\varepsilon_1 + M_1\frac{d^2}{h^2}(\varepsilon_1+\varepsilon_2) + M_2\frac{d^2}{h^2}\varepsilon_1] \\
M_2 &= \mathcal{O}\{\frac{\varepsilon_2}{h} + (\varepsilon_1+\varepsilon_2)\frac{d^3}{h^3}[m_2+\frac{h}{d}(M_1+M_2)]\}.
\end{aligned}$$

When θ_0 is sufficiently small we can eliminate m_2 from the right-hand sides of the above obtaining

$$m_1 = \mathcal{O}[\frac{\varepsilon_1}{d} + M_1 \frac{d}{h} (\varepsilon_1+\varepsilon_2) + M_2 \frac{d}{h} \varepsilon_1]$$

$$m_2 = \mathcal{O}\{\frac{\varepsilon_2}{d} + (\varepsilon_1+\varepsilon_2) \frac{d}{h} (M_1+M_2)\}$$

$$M_1 = \mathcal{O}[\frac{\varepsilon_1}{h} + M_1 \frac{d^2}{h^2} (\varepsilon_1+\varepsilon_2) + M_2 \frac{d^2}{h^2} \varepsilon_1]$$

$$M_2 = \mathcal{O}\{\frac{\varepsilon_2}{h} + (\varepsilon_1+\varepsilon_2) \frac{d^2}{h^2} (M_1+M_2)\}$$

When θ_0 is sufficiently small we can eliminate M_1 from the right-hand sides of the above to obtain

$$m_1 = \mathcal{O}(\frac{\varepsilon_1}{d} + M_2 \frac{d}{h} \varepsilon_1)$$

$$m_2 = \mathcal{O}[\frac{\varepsilon_2}{d} + (\varepsilon_1+\varepsilon_2) \frac{d}{h} M_2]$$

$$M_1 = \mathcal{O}(\frac{\varepsilon_1}{d} + M_2 \frac{d^2}{h^2} \varepsilon_1)$$

$$M_2 = \mathcal{O}[\frac{\varepsilon_2}{h} + (\varepsilon_1+\varepsilon_2) \frac{d^2}{h^2} M_2] .$$

For θ_0 sufficiently small M_2 can be eliminated from the right-hand sides of the above to obtain

$$\begin{aligned} m_\omega &= \mathcal{O}(\frac{\varepsilon_\omega}{d}) \\ M_\omega &= \mathcal{O}(\frac{\varepsilon_\omega}{h}) \end{aligned} \tag{3.28}$$

From (3.28) and (3.26) we get (3.9) for $\ell = 3$.

As we continue proving (3.9) for larger values of ℓ, we will at the same time prove

$$\begin{aligned} |a_{k\omega}| &= \mathcal{O}(\frac{\varepsilon_\omega}{d^k}) \\ |b_{k\omega}| &= \mathcal{O}(\frac{\varepsilon_\omega}{hd^{k-1}}) \end{aligned} \tag{3.29}$$

which holds for $k = 1,2,3,\ldots$ at points where $\zeta_{k+3} \neq 0$.

Since $1/\zeta_3 \leq 16/9$ at points where $\zeta_4 \neq 0$, (3.28) implies that $|\dot{t}_\omega| = \mathcal{O}(\varepsilon_\omega/d)$ and $|t'_\omega| = \mathcal{O}(\varepsilon_\omega/h)$ at points where $\zeta_4 \neq 0$.

From (2.5) for $k = 1$

$$a_{11} = \dot{t}_1 + [(t_1+t_2)t'_1+t_1t'_2]F(t_1,t_2)$$

$$a_{12} = \dot{t}_2 + (t_1+t_2)(t'_1+t'_2)F(t_1,t_2)$$

and by definition

$$b_{1\omega} = t'_\omega \ .$$

Therefore

$$|a_{11}| = \mathcal{O}(|\dot{t}_1| + \theta^2|t'_1| + \varepsilon_1|t'_2|) = \mathcal{O}(\frac{\varepsilon_1}{d})$$

$$|a_{12}| = \mathcal{O}[|\dot{t}_2| + (\varepsilon_1+\varepsilon_2)(|t'_1|+|t'_2|)] = \mathcal{O}(\frac{\varepsilon_2}{d})$$

$$|b_{1\omega}| = \mathcal{O}(\frac{\varepsilon_\omega}{h})$$

at points where $\zeta_4 \neq 0$.

Therefore (3.29) holds for $k = 1$.

Now we change the meanings of m_ω and M_ω to

$$\begin{aligned} m_\omega &= \max \zeta_4|\ddot{t}_\omega| \\ M_\omega &= \max \zeta_4|t''_\omega| \end{aligned} \tag{3.20}$$

From (3.20)

$$([(t_1+t_2)t'_1+t_1t'_2]F(t_1,t_2))^{\dddot{}}$$

$$= [(t_1+t_2)t_1^{\prime(4)}+t_1t_2^{\prime(4)}+(\dot{t}_1+\dot{t}_2)t'''_1+\dot{t}_1t'''_2+(\ddot{t}_1+\ddot{t}_2)t''_1+\ddot{t}_1t''_2$$

$$+(\dddot{t}_1 + \dddot{t}_2)t'_1+\dddot{t}_1t'_2+(t'_1+t'_2)^2t''_1+(t'_1+t'_2)t''_2\ t'_1$$

$$+ t_1(t'_2t'''_2+t''^2_2+t'^2_2t''_2+t'^4_2) + (t'_1+t'_2)^3t'_1]F(t_1,t_2)$$

$$[t_1(t_1+t_2)F(t_1,t_2)]^{\prime(4)} =$$

$$\begin{aligned} &= [(t_1+t_2)t_1^{\prime(4)} + t_1t_2^{\prime(4)} + (t_1'+t_2')t_1''' + t_1't_2''' + (t_1''+t_2'')t_1'' \\ &\quad + (t_1'+t_2')^2t_1'' + (t_1'+t_2')t_1't_2'' + t_1'(t_1'+t_2')^3 \\ &\quad + t_1(t_2''^2 + t_2't_2''' + t_2'^2t_2'' + t_2'^4)]F(t_1,t_2) \end{aligned} \tag{3.31}$$

$$\begin{aligned} &[(t_1+t_2)(t_1'+t_2')F(t_1,t_2)]^{\dddot{}} \\ &= [(t_1+t_2)(t_1^{\prime(4)}+t_2^{\prime(4)}) + (\dot t_1+\dot t_2)(t_1'''+t_2''') + (\ddot t_1+\ddot t_2)(t_1''+t_2'') \\ &\quad + (t_1'+t_2')(\dddot t_1 + \dddot t_2) + (t_1'+t_2')^2(t_1''+t_2'') + (t_1'+t_2')^4]F(t_1,t_2) \end{aligned}$$

$$\begin{aligned} &[(t_1^2+t_2^2)F(t_1,t_2)]^{\prime(4)} \\ &= [(t_1+t_2)(t_1^{\prime(4)}+t_2^{\prime(4)}) + (t_1'+t_2')(t_1'''+t_2''') + (t_1''+t_2'')^2 \\ &\quad + (t_1'+t_2')^2(t_1''+t_2'') + (t_1'+t_2')^4]F(t_1,t_2) . \end{aligned}$$

Since $\zeta_4 = \mathcal{O}(\zeta_4\zeta_k)$ for $k = 1,2,3$, (3.31) gives

$$\begin{aligned} &\zeta_4|([(t_1+t_2)t_1'+t_1t_2']F(t_1,t_2))^{\dddot{}}| \\ &= \mathcal{O}[\theta^2\zeta_4|t_1^{\prime(4)}| + \varepsilon_1\zeta_4|t_2^{\prime(4)}| + \frac{1}{d}(\varepsilon_1+\varepsilon_2)\zeta_3|t_1'''| + \frac{\varepsilon_1}{d}\zeta_3|t_2'''| \\ &\quad + M_1\zeta_2(|\ddot t_1|+|\ddot t_2|) + M_2\zeta_2|\ddot t_1| + \frac{\varepsilon_1}{h}\zeta_3(|\dddot t_1| + |\dddot t_2|) \\ &\quad + \frac{\varepsilon_2}{h}\zeta_3|\dddot t_1| + \frac{1}{h^2}(\varepsilon_1+\varepsilon_2)^2\zeta_2|t_1''| + \frac{\varepsilon_1}{h^2}(\varepsilon_1+\varepsilon_2)\zeta_2|t_2''| \\ &\quad + \frac{\varepsilon_1\varepsilon_2}{h}\zeta_2|t'''| + \varepsilon_1M_2\zeta_2|t_2''| + \frac{\varepsilon_1\varepsilon_2^2}{h^2}\zeta_2|t_2''| + \frac{\varepsilon_1\varepsilon_2^4}{h^4} + \frac{\varepsilon_1}{h^4}(\varepsilon_1+\varepsilon_2)^3] \end{aligned}$$

$$\begin{aligned} &\zeta_4|[t_1(t_1+t_2)F(t_1,t_2)]^{\prime(4)}| \\ &= [\theta^2\zeta_4|t_1^{\prime(4)}| + \varepsilon_1\zeta_4|t_2^{\prime(4)}| + \frac{1}{h}(\varepsilon_1+\varepsilon_2)\zeta_3|t_1'''| + \frac{\varepsilon_1}{h}\zeta_3|t_2'''| \\ &\quad + M_1\zeta_2(|t_1''|+|t_2''|) + \frac{1}{h^2}(\varepsilon_1+\varepsilon_2)^2\zeta_2|t_1''| + \frac{\varepsilon_1}{h^2}(\varepsilon_1+\varepsilon_2)\zeta_2|t_2''| \\ &\quad + \frac{\varepsilon_1}{h^4}(\varepsilon_1+\varepsilon_2)^3 + \varepsilon_1M_2\zeta_2|t_2''| + \frac{\varepsilon_1\varepsilon_2}{h}\zeta_3|t_2'''| + \frac{\varepsilon_1\varepsilon_2^2}{h^2}\zeta_2|t_2''| + \frac{\varepsilon_1\varepsilon_2^4}{h^4}] \end{aligned}$$

$$\zeta_4 \left| [(t_1+t_2)(t_1'+t_2')F(t_1,t_2)]^{\cdots} \right|$$

$$= \mathcal{O}[(\varepsilon_1+\varepsilon_2)\zeta_4 |t_1'^{(4)}| + \theta^2 \zeta_4 |t_2'^{(4)}| + \frac{1}{d} (\varepsilon_1+\varepsilon_2)\zeta_3(|t_1'''|+|t_2'''|)$$

$$+ (M_1+M_2)\zeta_2(|\ddot{t}_1|+|\ddot{t}_2|) + \frac{1}{h}(\varepsilon_1+\varepsilon_2)\zeta_3(|\dddot{t}_1|+|\dddot{t}_2|)$$

$$+ \frac{1}{h^2}(\varepsilon_1+\varepsilon_2)^2\zeta_2(|t_1''|+|t_2''|) + \frac{1}{h^4}(\varepsilon_1+\varepsilon_2)^4]$$

$$\zeta_4 \left| [(t_1^2+t_2^2)F(t_1,t_2)]'^{(4)} \right|$$

$$= \mathcal{O}[(\varepsilon_1+\varepsilon_2)\zeta_4 |t_1'^{(4)}| + \theta^2 \zeta_4 |t_2'^{(4)}| + \frac{1}{h} (\varepsilon_1+\varepsilon_2)\zeta_3(|t_1'''|+|t_2'''|)$$

$$+ (M_1+M_2)\zeta_2(|t_1''|+|t_2''|) + \frac{1}{h^2} (\varepsilon_1+\varepsilon_2)^2\zeta_2(|t_1''|+|t_2''|) + \frac{1}{h^4}(\varepsilon_1+\varepsilon_2)^4].$$

Using the above and (3.9) for $\ell < 4$

$$\begin{gathered}
||\zeta_4([t_1+t_2)t_1'+t_1t_2']F(t_1,t_2))^{\cdots}||_4 \\
= [\theta^2 B_{41} + \varepsilon_1 B_{42} + \frac{\sqrt{h}}{d} (\theta^2 M_1 + \varepsilon_1 M_2) + \frac{h\sqrt{h}}{d^4} \varepsilon_1] \\
||\zeta_4[t_1(t_1+t_2)F(t_1,t_2)]'^{(4)}||_4 \\
= \mathcal{O}[\theta^2 B_{41}+\varepsilon_1 B_{42} + \frac{1}{\sqrt{h}} (\theta^2 M_1+\varepsilon_1 M_2) + \frac{\sqrt{h}}{d^3} \varepsilon_1] \\
||\zeta_4[(t_1+t_2)(t_1'+t_2')F(t_1,t_2)]^{\cdots}||_4 \\
= \mathcal{O}[(\varepsilon_1+\varepsilon_2)B_{41}+\theta^2 B_{42}+ \frac{\sqrt{h}}{d} (\varepsilon_1+\varepsilon_2)(M_1+M_2)+ \frac{h\sqrt{h}}{d^4} \varepsilon_2] \\
||\zeta_4[(t_1^2+t_2^2)F(t_1,t_2)]'^{(4)}||_4 \\
= \mathcal{O}[(\varepsilon_1+\varepsilon_2)B_{41}+\theta^2 B_{42}+ \frac{1}{\sqrt{h}} (\varepsilon_1+\varepsilon_2)(M_1+M_1)+ \frac{\sqrt{h}}{d^3} \varepsilon_2]
\end{gathered} \tag{3.32}$$

Also

$$||\ddot{t}_\omega||_4 = \mathcal{O}(||\zeta_2\ddot{t}_\omega||_2) = \mathcal{O}(A_{2\omega}) = \mathcal{O}(\frac{\varepsilon_\omega\sqrt{\hbar}}{d})$$

$$||[t_1(t_1+t_2)F(t_1,t_2)]''||_4 = \mathcal{O}(||\zeta_2[t_1(t_1+t_2)F(t_1,t_2)]''||_2) = \mathcal{O}(\frac{\sqrt{\hbar}\varepsilon_1}{d})$$

$$||[(t_1^2+t_2^2)F(t_1,t_2)]''||_4 = \mathcal{O}(||\zeta_2[(t_1^2+t_2^2)F(t_1,t_2)]''||_2) = \mathcal{O}(\frac{\sqrt{\hbar}\varepsilon_2}{d})$$

where we have used (3.13) in the last two lines.

Substituting the above and (3.32) into (3.7) and (3.8)

$$\begin{aligned}
A_{41} &= \mathcal{O}[\frac{\varepsilon_1\sqrt{\hbar}}{d^3} + \theta_0\theta B_{41} + \frac{d}{\hbar}\,\varepsilon_1 B_{42} + \frac{1}{\sqrt{\hbar}}\,(\theta^2 M_1+\varepsilon_1 M_2)] + o(\frac{\hbar}{d}\,B_{41}) \\
A_{42} &= \mathcal{O}[\frac{\varepsilon_2\sqrt{\hbar}}{d^3} + \frac{d}{\hbar}\,(\varepsilon_1+\varepsilon_2)B_{41} + \theta_0\theta B_{42} + \frac{1}{\sqrt{\hbar}}\,(\varepsilon_1+\varepsilon_2)(M_1+M_2)] \\
&\qquad + o(\frac{\hbar}{d}\,B_{42}) \\
B_{41} &= \mathcal{O}[\frac{\varepsilon_1}{d^2\sqrt{\hbar}} + \frac{d}{\hbar}\,A_{41} + \theta_0^2 B_{41} + \varepsilon_1 B_{42} + \frac{1}{\sqrt{\hbar}}\,(\theta^2 M_1+\varepsilon_1 M_2)] \\
B_{42} &= \mathcal{O}[\frac{\varepsilon_2}{d^2\sqrt{\hbar}} + \frac{d}{\hbar}\,A_{42} + (\varepsilon_1+\varepsilon_2)B_{41} + \theta_0^2 B_{42} + \frac{1}{\sqrt{\hbar}}\,(\varepsilon_1+\varepsilon_2)(M_1+M_2)]
\end{aligned} \tag{3.33}$$

We can "solve" (3.33) in the usual way if θ_0 is sufficiently small, obtaining

$$\begin{aligned}
A_{41} &= \mathcal{O}[\frac{\varepsilon_1\sqrt{\hbar}}{d^3} + \frac{1}{\sqrt{\hbar}}\,(\theta^2 M_1+\varepsilon_1 M_2)] \\
A_{42} &= \mathcal{O}[\frac{\varepsilon_2\sqrt{\hbar}}{d^3} + \frac{1}{\sqrt{\hbar}}\,(\varepsilon_1+\varepsilon_2)(M_1+M_2)] \\
B_{41} &= \mathcal{O}[\frac{\varepsilon_1}{d^2\sqrt{\hbar}} + \frac{d}{\hbar\sqrt{\hbar}}\,(\theta^2 M_1+\varepsilon_1 M_2)] \\
B_{42} &= \mathcal{O}[\frac{\varepsilon_2}{d^2\sqrt{\hbar}} + \frac{d}{\hbar\sqrt{\hbar}}\,(\varepsilon_1+\varepsilon_2)(M_1+M_2)]
\end{aligned} \tag{3.34}$$

From $(3.4)_{11}$

$$\frac{\sqrt{\bar{h}}}{d}\zeta_4|\ddot{t}_\omega| = \mathcal{O}[||\zeta_4\dot{t}_\omega^{(4)}||_4 + \frac{1}{d^2}||\ddot{t}_\omega||_4 + \frac{h^2}{d^2}||\zeta_4 t_\omega'^{(4)}||_4] \quad .$$

$$= \mathcal{O}(A_{4\omega} + \frac{1}{d^2}||\zeta_2\ddot{t}_\omega||_2 + \frac{h^2}{d^2}B_{4\omega})$$

$$= \mathcal{O}(\frac{\varepsilon_\omega\sqrt{\bar{h}}}{d^3} + A_{4\omega} + \frac{h^2}{d^2}B_{4\omega}) \quad ,$$

and

$$\frac{\sqrt{\bar{h}}}{d}\zeta_4|t_\omega''| = \mathcal{O}[||\zeta_4 t_\omega'^{(4)}||_4 + \frac{1}{d^2}||t''||_4]$$

$$= \mathcal{O}[B_{4\omega} + \frac{1}{d^2}||\zeta_2 t_\omega''||_2]$$

$$= \mathcal{O}(B_{4\omega} + \frac{\varepsilon_\omega}{d^2\sqrt{\bar{h}}})$$

so that

$$m_\omega = \mathcal{O}(\frac{\varepsilon_\omega}{d^2} + \frac{d}{\sqrt{\bar{h}}}A_{4\omega} + \frac{h\sqrt{\bar{h}}}{d}B_{4\omega})$$

$$M_\omega = \mathcal{O}(\frac{\varepsilon_\omega}{d\bar{h}} + \frac{d}{\sqrt{\bar{h}}}B_{4\omega}) \quad .$$

From the above and (3.34),

$$\begin{aligned} m_1 &= \mathcal{O}[\frac{\varepsilon_1}{d^2} + \frac{d}{\bar{h}}(\theta^2 M_1 + \varepsilon_1 M_2)] \\ m_2 &= \mathcal{O}[\frac{\varepsilon_2}{d^2} + \frac{d}{\bar{h}}(\varepsilon_1+\varepsilon_2)(M_1+M_2)] \\ M_1 &= \mathcal{O}[\frac{\varepsilon_1}{d\bar{h}} + \frac{d^2}{h^2}(\theta^2 M_1+\varepsilon_1 M_2)] \\ M_2 &= \mathcal{O}[\frac{\varepsilon_2}{d\bar{h}} + \frac{d^2}{h^2}(\varepsilon_1+\varepsilon_2)(M_1+M_2)] \end{aligned} \tag{3.35}$$

We can "solve" (3.35) when θ_0 is sufficiently small to obtain

$$\begin{aligned} m_\omega &= \mathcal{O}(\frac{\varepsilon_\omega}{d^2}) \\ M_\omega &= \mathcal{O}(\frac{\varepsilon_\omega}{\bar{h}d}) \end{aligned} \tag{3.36}$$

If we substitute (3.36) into (3.34), we get (3.9) for $\ell=4$. Also from (3.36)

$$|\ddot{t}_\omega| = \mathcal{O}(\frac{\varepsilon_\omega}{d^2})$$

$$|t''_\omega| = \mathcal{O}(\frac{\varepsilon_\omega}{hd})$$

at points where $\zeta_5 \neq 0$.

From the above, (2.5) for k = 2, and (3.12)

$$|a_{2\omega}| = \mathcal{O}(\frac{\varepsilon_\omega}{d^2}) \quad \text{where} \quad \zeta_5 \neq 0.$$

By definition

$$|b_{2\omega}| = |t''_\omega| = \mathcal{O}(\frac{\varepsilon_\omega}{hd}) \quad \text{where } \zeta_5 \neq 0.$$

Thus (3.29) is valid for k = 2.

When we differentiate (3.31), we see that the result is linear in those derivatives for which we do not have pointwise estimates, namely those of order 3, 4, and 5. This makes the next step easier and from (3.31) we obtain

$$\begin{aligned}
&||\zeta_5([(t_1+t_2)t_1'+t_1t_2']F(t_1,t_2))^{\cdot(4)}||_5 \\
&= \mathcal{O}(\theta^2 B_{51} + \varepsilon_1 B_{52} + \frac{h\sqrt{h}}{d^5}\varepsilon_1) \\
&||\zeta_5[t_1(t_1+t_2)F(t_1,t_2)]^{'(5)}||_5 \\
&= \mathcal{O}(\theta^2 B_{51} + \varepsilon_1 B_{52} + \frac{\sqrt{h}}{d^4}\varepsilon_1) \\
&||\zeta_5[(t_1+t_2)(t_1'+t_2')F(t_1,t_2)]^{\cdot(4)}||_5 \\
&= \mathcal{O}[(\varepsilon_1+\varepsilon_2)B_{51}+\theta^2 B_{52} + \frac{h\sqrt{h}}{d^5}\varepsilon_2] \\
&||\zeta_5[(t_1^2+t_2^2)F(t_1,t_2)]^{'(5)}|| = \mathcal{O}[(\varepsilon_1+\varepsilon_2)B_{51}+\theta^2 B_{52}+\frac{\sqrt{h}}{d^4}\varepsilon_2]
\end{aligned} \tag{3.37}$$

From (3.7), (3.8), (3.20), and (3.37)

$$(3.38)\quad \begin{aligned} A_{51} &= \mathcal{O}\left(\frac{\sqrt{h}}{d^4}\varepsilon_1 + \theta_0\theta B_{51} + \frac{d}{h}\varepsilon_1 B_{52}\right) + o\left(\frac{h}{d}B_{51}\right) \\ A_{52} &= \mathcal{O}\left(\frac{\sqrt{h}}{d^4}\varepsilon_2 + \frac{d}{h}(\varepsilon_1+\varepsilon_2)B_{51} + \theta_0\theta B_{52}\right) + o\left(\frac{h}{d}B_{52}\right) \\ B_{51} &= \mathcal{O}\left(\frac{\varepsilon_1}{\sqrt{h}d^3} + \frac{d}{h}A_{51} + \theta_0^2 B_{51} + \varepsilon_1 B_{52}\right) \\ B_{52} &= \mathcal{O}\left[\frac{\varepsilon_2}{\sqrt{h}d^3} + \frac{d}{h}A_{52} + (\varepsilon_1+\varepsilon_2)B_{51} + \theta_0^2 B_{52}\right] \end{aligned}$$

From (3.38) we get (3.9) for $\ell = 5$ when θ_0 is sufficiently small in the usual way. Then $(3.4)_{11}$ gives

$$\frac{\sqrt{h}}{d}\zeta_5|\ddot{t}_\omega| = \mathcal{O}\left(A_{5\omega} + \frac{1}{d^2}A_{3\omega} + \frac{h^2}{d^2}B_{5\omega}\right) = \mathcal{O}\left(\frac{\sqrt{h}\varepsilon_\omega}{d^4}\right)$$

$$\frac{\sqrt{h}}{d}\zeta_5|t'''_\omega| = \mathcal{O}\left(B_{5\omega} + \frac{1}{d^2}B_{3\omega}\right) = \mathcal{O}\left(\frac{\varepsilon_\omega}{\sqrt{h}d^4}\right)$$

which imply $|\dddot{t}_\omega| = \mathcal{O}(\varepsilon_\omega/d^3)$ and $|t'''_\omega| = \mathcal{O}(\varepsilon_\omega/hd^2)$ at points where $\zeta_6 \neq 0$. This with (2.5) gives (3.29) for $k = 3$.

It is easy to see that this procedure can be continued to complete the proof of (3.9) and (3.29).

This completes the proof of $(0.10)_1$.

Since $\partial^k t_{\alpha 31}/\partial x_1^{k_1}\partial x_2^{k_2} = 0$ for $x_3 = h$, by the mean value theorem

$$\frac{\partial^k t_{\alpha 31}}{\partial x_1^{k_1}\partial x_2^{k_2}} = \mathcal{O}\left(h \max_{|x_3|\leq h}\left|\frac{\partial^{k+1} t_{\alpha 31}}{\partial x_1^{k_1}\partial x_2^{k_2}\,\partial x_3}\right|\right) = \mathcal{O}\left(\frac{h\,\varepsilon_1}{d^{k+1}}\right)$$

using $(0.10)_1$. Thus $(0.10)_2$ is verified.

Next $\partial^k t_{\alpha 32}/\partial x_1^{k_1}\partial x_2^{k_2}$ is an odd function of x_3 and is therefore zero for $x_3 = 0$. Since also it is zero for $x_3 = \pm h$, it has at least three zeros on each vertical line. Therefore, using the mean value theorem several times, we have

$$\frac{\partial^k t_{\alpha 32}}{\partial x_1^{k_1} \partial x_2^{k_2}} = \mathcal{O}\left(h^3 \max_{|x_3| \leq h} \left| \frac{\partial^{k+3} t_{\alpha 32}}{\partial x_1^{k_1} \partial x_2^{k_2} \partial x_3^3} \right| \right) = \mathcal{O}\left(\frac{h^3 \varepsilon_2}{d^{k+3}}\right)$$

from $(0.10)_1$. This gives $(0.10)_3$ for $k_3 = 0$. Similarly $(0.10)_3$ is true for $k_3 = 1,2$.

By more reasoning of the above type, the remainder of (0.10) can be proved.

From (1.13) and $(0.10)_1$

$$z_1 = (t_1^2 + t_1 t_2) F(t_1,t_2) = \mathcal{O}(\theta^2 \varepsilon_1)$$

$$z_2 = (t_1^2 + t_2^2) F(t_1,t_2) = \mathcal{O}(\theta^2 \varepsilon_2)$$

$$z_1' = [(t_1+t_2)t_1' + t_1 t_2'] F(t_1,t_2) = \mathcal{O}(\frac{\theta^2}{h} \varepsilon_1)$$

$$z_2' = (t_1+t_2)(t_1'+t_2') F(t_1,t_2) = \mathcal{O}(\frac{\theta^2}{h} \varepsilon_2)$$

$$z_1'' = [(t_1+t_2)t_1'' + t_1 t_2'' + (t_1'+t_2')t_1' + t_1 t_2'^2] F(t_1,t_2) = \mathcal{O}(\frac{\theta^2}{h^2} \varepsilon_1)$$

$$z_2'' = [(t_1+t_2)(t_1''+t_2'') + (t_1'+t_2')^2] F(t_1,t_2) = \mathcal{O}(\frac{\theta^2}{h^2} \varepsilon_2) \ .$$

It is clear that each additional differentiation will result in a multiplication of the estimate by 1/d. Therefore we have

$$z_\omega = \mathcal{O}(\theta^2 \varepsilon_\omega) = \mathcal{O}(\frac{h^2}{d^2} \varepsilon_\omega)$$

$$(3.39) \qquad z_\omega' = \mathcal{O}(\frac{\theta^2}{h} \varepsilon_\omega) = \mathcal{O}(\frac{h}{d^2} \varepsilon_\omega)$$

$$z_\omega'^{(k)} = \mathcal{O}(\frac{\theta^2}{h^2 d^{k-2}} \varepsilon_\omega) = \mathcal{O}(\frac{\varepsilon_\omega}{d^k}) \quad \text{for } k = 2,3,\ldots \ .$$

Since $r_{ij\omega}$ is a linear combination from the set z_ω (see the line before (1.14)), (3.39) gives

$$
\begin{aligned}
& r_{\omega} &&= \mathcal{O}(\theta^{2}\varepsilon_{\omega}) &&= \mathcal{O}(\frac{h^{2}}{d^{2}}\,\varepsilon_{\omega}) \\
(3.40)\qquad & r'_{\omega} &&= \mathcal{O}(\frac{\theta^{2}}{h}\varepsilon_{\omega}) &&= \mathcal{O}(\frac{h}{d^{2}}\,\varepsilon_{\omega}) \\
& r'^{(k)}_{\omega} &&= \mathcal{O}(\frac{\theta^{2}}{h^{2}d^{k-2}}\,\varepsilon_{\omega}) &&= \mathcal{O}(\frac{\varepsilon_{\omega}}{d^{k}}) \quad \text{for } k = 2,3,\ldots\ .
\end{aligned}
$$

From $(1.9)_5$, $(0.10)_1$, and (3.40) we now obtain $(0.11)_1$.

From (1.6) we see that for $x_3 = \pm h$,

$$0 = \frac{2\mu}{E}\, e_{\alpha 3} + z_{\alpha 3} = [\frac{2\mu}{E}\,\delta_{\alpha\beta} + \delta_{\alpha\beta} eF(e) + e_{\alpha\beta}F(e)]e_{\beta 3}\ .$$

By the implicit function theorem, these equations can be solved for $e_{\alpha 3}$ in terms of the other strains when the strains are sufficiently near zero (i.e. when θ_0 is small enough). Since $e_{\alpha 3} = 0$ is a solution, it is also the unique solution. Thus $e_{\alpha 3\omega} = 0$ for $x_3 = \pm h$ and $(0.11)_2$ and $(0.11)_3$ are now proved in the same away as $(0.10)_2$ and $(0.10)_3$. $(0.10)_4$ is proved as was $(0.10)_5$.

This completes the proof of (0.10) and (0.11).

4. Proof of (0.14), (0.15), (0.16) and (0.17).

For any function $A = A(x_1,x_2,x_3)$ we introduce the notation A_0, $(A)_0$, $[A]_0$, or $\{A\}_0$ for $A(x_1,x_2,0)$. Then from (0.2)

$$\sigma_{\alpha\beta} = (e_{\alpha\beta})_0 = (e_{\alpha\beta 2})_0$$

$$b_{\alpha\beta} = (e_{\alpha\beta,3})_0 = (e_{\alpha\beta 1,3})_0 .$$

(0.16) follows from these and (0.11).

To prove (0.16), (0.17), and (0.19), we will calculate estimates for the derivatives of several orders of the stresses and strains with respect to x_3 evaluated on the middle surface. Most of the rest of the report will be devoted to this.

From (1.6)

$$t_{33} = \frac{\lambda}{E} e_{\gamma\gamma} + \frac{\lambda+2\mu}{E} e_{33} + z_{33}$$

so that

$$(e_{33})_0 = \frac{E}{\lambda+2\mu} (t_{332}-z_{332})_0 - \frac{\lambda}{\lambda+2\mu} \sigma_{\gamma\gamma} = - \frac{\lambda}{\lambda+2\mu} \sigma_{\gamma\gamma} + \mathcal{O}(\frac{h^2}{d^2} \varepsilon_2)$$

using (0.10) and (3.39).

Similarly

$$(e_{33,3})_0 = \frac{E}{\lambda+2\mu} (t_{331,3}-z_{331,3})_0 - \frac{\lambda}{\lambda+2\mu} b_{\gamma\gamma} = - \frac{\lambda}{\lambda+2\mu} b_{\gamma\gamma} + \mathcal{O}(\frac{h}{d^2} \varepsilon_1) .$$

Using the above in (1.6),

$$(t_{\alpha\beta})_0 = \frac{\lambda}{E} (e_{kk})_0 \delta_{\alpha\beta} + \frac{2\mu}{E} (e_{\alpha\beta})_0 + (z_{\alpha\beta 2})_0$$

$$= \frac{2\mu}{E} (\frac{\lambda}{\lambda+2\mu} \sigma_{\gamma\gamma}\delta_{\alpha\beta} + \sigma_{\alpha\beta}) + \mathcal{O}(\frac{h^2}{d^2} \varepsilon_2)$$

$$(t_{\alpha\beta,3})_0 = \frac{\lambda}{E} (e_{kk,3})_0 \delta_{\alpha\beta} + \frac{2\mu}{E} (e_{\alpha\beta,3})_0 + (z_{\alpha\beta 1,3})_0$$

$$= \frac{2\mu}{E} (\frac{\lambda}{\lambda+2\mu} b_{\gamma\gamma}\delta_{\alpha\beta} + b_{\alpha\beta}) + \mathcal{O}(\frac{h}{d^2} \varepsilon_1) .$$

We list these as

$$\begin{aligned}
(e_{33})_0 &= -\frac{\lambda}{\lambda+2\mu}\sigma_{\gamma\gamma} + \mathcal{O}(\frac{h^2}{d^2}\varepsilon_2) \\
(e_{33,3})_0 &= -\frac{\lambda}{\lambda+2\mu}b_{\gamma\gamma} + \mathcal{O}(\frac{h}{d^2}\varepsilon_1) \\
(t_{\alpha\beta})_0 &= \frac{2\mu}{E}(\frac{\lambda}{\lambda+2\mu}\sigma_{\gamma\gamma}\delta_{\alpha\beta} + \sigma_{\alpha\beta}) + \mathcal{O}(\frac{h^2}{d^2}\varepsilon_2) \\
(t_{\alpha\beta,3})_0 &= \frac{2\mu}{E}(\frac{\lambda}{\lambda+2\mu}b_{\gamma\gamma}\delta_{\alpha\beta} + b_{\alpha\beta}) + \mathcal{O}(\frac{h}{d^2}\varepsilon_1)
\end{aligned}\tag{4.1}$$

Using $(4.1)_1$, $(4.1)_2$, (0.11), and (0.12)

$$(s_1)_0 = (e_{ii})_0 = \frac{2\mu}{\lambda+2\mu}\sigma_{\gamma\gamma} + \mathcal{O}(\frac{h^2}{d^2}\varepsilon_2)$$

$$(s_{1,3})_0 = (e_{ii,3})_0 = \frac{2\mu}{\lambda+2\mu}b_{\gamma\gamma} + \mathcal{O}(\frac{h}{d^2}\varepsilon_1)$$

$$(s_{1,33})_0 = (e_{ii,33})_0 = \mathcal{O}(\frac{\varepsilon_2}{d^2})$$

$$(s_{1,333})_0 = (e_{ii,333})_0 = \mathcal{O}(\frac{\varepsilon_1}{hd^2})$$

$$(s_{1,3333})_0 = (e_{ii,3333})_0 = \mathcal{O}(\frac{\varepsilon_2}{d^4})$$

$$(s_{1,33333})_0 = (e_{ii,33333})_0 = \mathcal{O}(\frac{\varepsilon_1}{hd^4})$$

$$(s_2)_0 = (e_{ij}e_{ij})_0 = \sigma_{\gamma\nu}\sigma_{\gamma\nu} + \frac{\lambda^2}{(\lambda+2\mu)^2}\sigma_{\gamma\gamma}\sigma_{\nu\nu} + \mathcal{O}(\frac{h^2}{d^2}\theta^2\varepsilon_2)$$

$$(s_{2,3})_0 = 2(e_{ij}e_{ij,3})_0 = 2\sigma_{\gamma\nu}b_{\gamma\nu} + \frac{2\lambda^2}{(\lambda+2\mu)^2}\sigma_{\gamma\gamma}b_{\nu\nu} + \mathcal{O}(\frac{h}{d^2}\varepsilon_1\varepsilon_2)$$

$$\begin{aligned}
(s_{2,33})_0 &= 2(e_{ij}e_{ij,33} + e_{ij,3}e_{ij,3})_0 \\
&= 2b_{\gamma\nu}b_{\gamma\nu} + \frac{2\lambda^2}{(\lambda+2\mu)^2}b_{\gamma\gamma}b_{\nu\nu} + \mathcal{O}(\frac{\theta^2}{d^2}\varepsilon_2)
\end{aligned}\tag{4.2}$$

$$(s_{2,333})_0 = 2(e_{ij}e_{ij,333} + 3e_{ij,3}e_{ij,33})_0 = \mathcal{O}(\frac{\varepsilon_1\varepsilon_2}{hd^2})$$

$$(s_{2,3333})_0 = 2(e_{ij}e_{ij,3333} + 4e_{ij,3}e_{ij,333} + 3e_{ij,33}e_{ij,33}) = \mathcal{O}(\frac{\theta^2}{h^2d^2}\varepsilon_2)$$

$$\begin{aligned}
(s_{2,33333})_0 &= 2(e_{ij}e_{ij,33333} + 5e_{ij,3}e_{ij,3333} + 10e_{ij,33}e_{ij,333})_0 \\
&= \mathcal{O}(\frac{\varepsilon_1\varepsilon_2}{hd^4})
\end{aligned}$$

$$(s_3)_0 = (e_{ij}e_{jk}e_{ki})_0 = \mathcal{O}(\theta^2\varepsilon_2^2)$$

$$(s_{3,3})_0 = 3(e_{ij}e_{jk}e_{ki,3})_0 = \mathcal{O}(\frac{\theta^2}{h}\varepsilon_1\varepsilon_2)$$

$$(s_{3,33})_0 = 3(e_{ij}e_{jk}e_{ki,33}+2e_{ij}e_{jk,3}e_{ki,3})_0 = \mathcal{O}(\frac{\theta^2}{h^2}\varepsilon_2^2)$$

$$(s_{3,333})_0 = 3(e_{ij}e_{jk}e_{ki,333}+6e_{ij}e_{jk,3}e_{ki,33}+2e_{ij,3}e_{jk,3}e_{ki,3})_0$$
$$= \mathcal{O}(\frac{\theta^2}{h^3}\varepsilon_1\varepsilon_2)$$

$$(s_{3,3333})_0 = 3(e_{ij}e_{jk}e_{ki,3333}+8e_{ij}e_{jk,3}e_{ki,333}+6e_{ij}e_{jk,33}e_{ki,33}$$
$$+12e_{ij,3}e_{jk,3}e_{ki,33}) = \mathcal{O}(\frac{\theta^2\varepsilon_2^2}{h^2d^2})$$

$$(s_{3,33333})_0 = 3(e_{ij}e_{jk}e_{ki,33333}+ 10e_{ij}e_{jk,3}e_{ki,3333}$$
$$+ 20e_{ij}e_{jk,33}e_{ki,333}+20e_{ij,3}e_{jk,3}e_{ki,333}+30e_{ij,3}e_{jk,33}e_{ki,33})_0$$
$$= \mathcal{O}(\frac{\theta^2\varepsilon_1\varepsilon_2}{h^3d^2})$$

Let $\mathcal{A}_k$ denote any analytic function of s_1, s_2, s_3 of order k in the strains. Then

$$\mathcal{A}_1 = \sum_{k=1}^{3} \mathcal{B}_k s_k$$

$$\mathcal{A}_2 = \mathcal{B}_4 s_1^2 + \mathcal{B}_5 s_2 + \mathcal{B}_6 s_3$$

$$\mathcal{A}_3 = \mathcal{B}_7 s_1^3 + \mathcal{B}_8 s_1 s_2 + \mathcal{B}_9 s_3 + \mathcal{B}_{10} s_2^2$$

where $\mathcal{B}_k$ (k=1,...,10) are analytic functions of s_1, s_2, s_3.

From the above and (4.2) we obtain

$$(\mathcal{A}_0)_0 = \mathcal{O}(1)$$
$$(\mathcal{A}_{0,3})_0 = \mathcal{O}(\frac{\varepsilon_1}{h})$$
$$(\mathcal{A}_{0,33})_0 = \mathcal{O}(\frac{\varepsilon_2}{d^2})$$
$$(\mathcal{A}_{0,333})_0 = \mathcal{O}(\frac{\varepsilon_1}{hd^2})$$
$$(\mathcal{A}_{0,3333})_0 = \mathcal{O}(\frac{\varepsilon_2}{d^4})$$

$$
\begin{aligned}
(\mathcal{A}_1)_0 &= \mathcal{O}(\varepsilon_2)\\
(\mathcal{A}_{1,3})_0 &= \mathcal{O}(\tfrac{\varepsilon_1}{h})\\
(\mathcal{A}_{1,33})_0 &= \mathcal{O}(\tfrac{\varepsilon_2}{d^2})\\
(\mathcal{A}_{1,333})_0 &= \mathcal{O}(\tfrac{\varepsilon_1}{hd^2})\\
(\mathcal{A}_{1,3333})_0 &= \mathcal{O}(\tfrac{\varepsilon_2}{d^4})\\
(\mathcal{A}_2)_0 &= \mathcal{O}(\theta^2\varepsilon_2)\\
(\mathcal{A}_{2,3})_0 &= \mathcal{O}(\tfrac{\varepsilon_1\varepsilon_2}{h})\\
(\mathcal{A}_{2,33})_0 &= \mathcal{O}(\tfrac{\theta^2\varepsilon_2}{h^2})\\
(\mathcal{A}_{2,333})_0 &= \mathcal{O}(\tfrac{\varepsilon_1\varepsilon_2}{hd^2})\\
(\mathcal{A}_{2,3333})_0 &= \mathcal{O}(\tfrac{\theta^2\varepsilon_2}{h^2d^2})\\
(\mathcal{A}_{2,33333})_0 &= \mathcal{O}(\tfrac{\varepsilon_1\varepsilon_2}{hd^4})\\
(\mathcal{A}_3)_0 &= \mathcal{O}(\theta^2\varepsilon_2^2)\\
(\mathcal{A}_{3,3})_0 &= \mathcal{O}(\tfrac{\theta^2\varepsilon_1\varepsilon_2}{h})\\
(\mathcal{A}_{3,33})_0 &= \mathcal{O}(\tfrac{\theta^2\varepsilon_2^2}{h^2})\\
(\mathcal{A}_{3,333})_0 &= \mathcal{O}(\tfrac{\theta^2\varepsilon_1\varepsilon_2}{h^3})\\
(\mathcal{A}_{3,3333})_0 &= \mathcal{O}(\tfrac{\theta^2\varepsilon_2^2}{h^2d^2})
\end{aligned}
\tag{4.2A}
$$

From (1.1)

$$W = \frac{\lambda}{2} s_1^2 + \mu s_2 + \mathcal{A}_3 = \mathcal{A}_2.$$

This, (4.2), and (4.2A) give

$$
(4.3)\qquad
\begin{aligned}
(W)_0 &= \mu(\sigma_{\gamma\nu}\sigma_{\gamma\nu} + \frac{\lambda}{\lambda+2\mu}\,\sigma_{\gamma\gamma}\sigma_{\nu\nu}) + \mathcal{O}(\frac{h^2}{d^2}\,\theta^2\varepsilon_2)\\
(W_{,3})_0 &= 2\mu(\sigma_{\gamma\nu}b_{\gamma\nu} + \frac{\lambda}{\lambda+2\mu}\,\sigma_{\gamma\gamma}b_{\nu\nu}) + \mathcal{O}(\frac{h}{d^2}\,\varepsilon_1\varepsilon_2)\\
(W_{,33})_0 &= 2\mu(b_{\gamma\nu}b_{\gamma\nu} + \frac{\lambda}{\lambda+2\mu}\,b_{\gamma\gamma}b_{\nu\nu}) + \mathcal{O}(\frac{\theta^2}{d^2}\,\varepsilon_2)\\
(W_{,333})_0 &= \mathcal{O}(\frac{\varepsilon_1\varepsilon_2}{h\ d^2})\\
(W_{,3333})_0 &= \mathcal{O}(\frac{\theta^2\varepsilon_2}{h^2d^2})\\
(W_{,33333})_0 &= \mathcal{O}(\frac{\varepsilon_1\varepsilon_2}{hd^4})
\end{aligned}
$$

We let

$$
(4.4)\qquad W_{i0} = \frac{1}{E}\,W_i\Big|_{s_1=s_2=s_3=0} \qquad (i = 1,2,3,4)
$$

where W_1, W_2, W_3, W_4 are the functions of s_1, s_2, s_3 appearing in (1.1). Definitions (4.4) agree with (0.13) for i=1,2,4.

From $(1.6)_2$

$$
(4.5)\quad z_{ij} = (W_{20}s_2 + 3W_{40}s_1^2+\mathcal{A}_3)\,\delta_{ij} + 2e_{ij}(W_{20}s_1+\mathcal{A}_2)+3e_{ik}e_{kj}(W_{10}+\mathcal{A}_1) + 3e_{ik}e_{kj}(W_{10}+\mathcal{A}_1)
$$

where $\mathcal{A}_k$ (k=1,2,3) are analytic functions of s_1, s_2, s_3 of order k in the strains.

From (4.5), (4.2), (4.2A), and (0.11)

$$\begin{aligned}
(z_{\alpha\beta})_0 &= \delta_{\alpha\beta}(W_{20}\sigma_{\gamma\nu}\sigma_{\gamma\nu}+L_1\sigma_{\gamma\gamma}\sigma_{\nu\nu}) + L_2\,\sigma_{\alpha\beta}\sigma_{\gamma\gamma} + 3W_{10}\,\sigma_{\alpha\gamma}\sigma_{\gamma\beta} \\
&\quad + \mathcal{O}(\frac{h^2}{d^2}\,\theta^2\varepsilon_2) \\
(z_{\alpha\beta,3})_0 &= 2\delta_{\alpha\beta}(W_{20}\sigma_{\gamma\nu}b_{\gamma\nu} + L_1\sigma_{\gamma\gamma}b_{\nu\nu}) + L_2(\sigma_{\alpha\beta}b_{\gamma\gamma}+\sigma_{\gamma\gamma}b_{\alpha\beta}) \\
&\quad + 3W_{10}(\sigma_{\alpha\gamma}b_{\gamma\beta}+\sigma_{\beta\gamma}b_{\gamma\alpha}) + \mathcal{O}(\frac{h}{d^2}\,\varepsilon_1\varepsilon_2) \\
(z_{\alpha\beta,33})_0 &= 2\delta_{\alpha\beta}(W_{20}b_{\gamma\nu}b_{\gamma\nu}+L_1b_{\gamma\gamma}b_{\nu\nu}) + 2L_2\,b_{\alpha\beta}b_{\gamma\gamma} \\
&\quad + 6W_{10}b_{\alpha\gamma}b_{\gamma\beta} + \mathcal{O}(\frac{\theta^2}{d^2}\,\varepsilon_2) \\
(z_{\alpha\beta,333})_0 &= \mathcal{O}(\frac{\varepsilon_1\varepsilon_2}{hd^2}) \\
(z_{\alpha\beta,3333})_0 &= \mathcal{O}(\frac{\theta^2}{h^2d^2}\,\varepsilon_2)
\end{aligned}\tag{4.6}$$

Also from (4.5), (4.2), (4.1), and (0.11)

$$\begin{aligned}
(z_{33})_0 &= W_{20}\sigma_{\gamma\nu}\sigma_{\gamma\nu} + L_3\sigma_{\gamma\gamma}\sigma_{\nu\nu} + \mathcal{O}(\frac{h^2}{d^2}\,\theta^2\varepsilon_2) \\
(z_{33,3})_0 &= 2W_{20}\sigma_{\gamma\nu}b_{\gamma\nu} + 2L_3\sigma_{\gamma\gamma}b_{\nu\nu} + \mathcal{O}(\frac{h}{d^2}\,\varepsilon_1\varepsilon_2) \\
(z_{33,33})_0 &= 2W_{20}b_{\gamma\nu}b_{\gamma\nu} + 2L_3b_{\gamma\gamma}b_{\nu\nu} + \mathcal{O}(\frac{\theta^2}{d^2}\,\varepsilon_2) \\
(z_{33,333})_0 &= \mathcal{O}(\frac{\varepsilon_1\varepsilon_2}{hd^2}) \\
(z_{33,3333})_0 &= \mathcal{O}(\frac{\theta^2}{h^2d^2}\,\varepsilon_2)
\end{aligned}\tag{4.7}$$

where we recall that the constants L_k are defined in the Appendix.

From (1.16)

$$(g^{\alpha\beta})_0 = \delta_{\alpha\beta} + \mathcal{O}(\varepsilon_2)$$

$$(g^{\alpha 3})_0 = \mathcal{O}(\frac{h}{d}\varepsilon_1)$$

$$(g^{33})_0 = 1 + \mathcal{O}(\varepsilon_2) \quad .$$

From $(1.4)_1$

$$(\Gamma_{\alpha\beta\gamma})_0 = \sigma_{\beta\gamma,\alpha} + \sigma_{\alpha\gamma,\beta} - \sigma_{\alpha\beta,\gamma} = \mathcal{O}(\frac{\varepsilon_2}{d})$$

$$(\Gamma_{\alpha\beta 3})_0 = - b_{\alpha\beta} + \mathcal{O}(\frac{h}{d^2}\varepsilon_1) = \mathcal{O}(\frac{\varepsilon_1}{h})$$

$$(\Gamma_{\alpha 3\beta})_0 = b_{\alpha\beta} + \mathcal{O}(\frac{h}{d^2}\varepsilon_1) = \mathcal{O}(\frac{\varepsilon_1}{h})$$

$$(\Gamma_{\alpha 33})_0 = (e_{33,\alpha})_0 = - \frac{\lambda}{\lambda+2\mu}\sigma_{\gamma\gamma,\alpha} + \mathcal{O}(\frac{h^2}{d^3}\varepsilon_2) = \mathcal{O}(\frac{\varepsilon_2}{d})$$

$$(\Gamma_{33\alpha})_0 = \frac{\lambda}{\lambda+2\mu}\sigma_{\gamma\gamma,\alpha} + \mathcal{O}(\frac{h^2}{d^3}\varepsilon_2) = \mathcal{O}(\frac{\varepsilon_2}{d})$$

$$(\Gamma_{333})_0 = - \frac{\lambda}{\lambda+2\mu} b_{\gamma\gamma} + \mathcal{O}(\frac{h}{d^2}\varepsilon_1) = \mathcal{O}(\frac{\varepsilon_1}{h}) \quad .$$

Substituting the above into $(1.4)_3$, we obtain

$$(c_{\alpha\beta\gamma\nu})_0 = b_{\alpha\beta}b_{\gamma\nu} - b_{\alpha\gamma}b_{\beta\nu} + \mathcal{O}(\frac{\theta^2}{d^2}\varepsilon_2)$$

(4.8) $$(c_{\alpha\beta\gamma 3})_0 = \mathcal{O}(\frac{\varepsilon_1\varepsilon_2}{dh})$$

$$(c_{\alpha\beta 33})_0 = \frac{\lambda}{\lambda+2\mu} b_{\alpha\beta}b_{\gamma\gamma} - b_{\alpha\gamma}b_{\gamma\beta} + \mathcal{O}(\frac{\theta^2}{d^2}\varepsilon_2)$$

Using (4.8) in (1.5), we have

$$\sigma_{\beta\nu,\alpha\gamma} + \sigma_{\alpha\gamma,\beta\nu} - \sigma_{\alpha\beta,\gamma\nu} - \sigma_{\gamma\nu,\alpha\beta} = b_{\alpha\beta}b_{\gamma\nu} - b_{\alpha\gamma}b_{\beta\nu} + \mathcal{O}(\frac{\theta^2}{d^2}\varepsilon_2)$$

(4.9) $$b_{\alpha\gamma,\beta} - b_{\alpha\beta,\gamma} = \mathcal{O}(\frac{h}{d^3}\varepsilon_1)$$

$$(e_{\alpha\beta,33})_0 = \frac{\lambda}{\lambda+2\mu}(\sigma_{\gamma\gamma,\alpha\beta} - b_{\alpha\beta}b_{\gamma\gamma}) + b_{\alpha\gamma}b_{\gamma\beta} + \mathcal{O}(\frac{h^2}{d^4}\varepsilon_2) \quad .$$

There is only one independent equation in $(4.9)_1$. Hence we do not lose anything if we set $\alpha = \gamma$, $\beta = \nu$ and sum. This gives the one equation

$$(4.10) \qquad \sigma_{\nu\nu,\gamma\gamma} - \sigma_{\gamma\nu,\gamma\nu} = \frac{1}{2}\,(b_{\gamma\nu}b_{\gamma\nu} - b_{\gamma\gamma}b_{\nu\nu}) + \mathcal{O}(\frac{\theta^2}{d^2}\,\varepsilon_2) \; .$$

From (1.3)

$$(4.11) \qquad t_{ij,j} + 2\,e_{ik}\,t_{kj,j} + 2e_{ik,j}t_{kj} - \frac{1}{E}\,W_{,i} = 0 \; .$$

Letting $x_3 = 0$ in (4.11) and using (0.10), (0.11), and (4.3)

$$(4.12) \qquad \begin{aligned} (t_{\alpha j,j})_0 &= \mathcal{O}(\frac{\theta^2}{d}\,\varepsilon_2) \\ (t_{\alpha\gamma,\gamma})_0 &= \mathcal{O}(\frac{h^2}{d^3}\,\varepsilon_2) \\ (t_{3j,j})_0 &= \frac{2\mu}{E}\,(\sigma_{\gamma\nu}b_{\gamma\nu} + \frac{\lambda}{\lambda+2\mu}\,\sigma_{\gamma\gamma}b_{\nu\nu}) + \mathcal{O}(\frac{h}{d^2}\,\varepsilon_1\varepsilon_2) \\ &= \mathcal{O}(\frac{\varepsilon_1\varepsilon_2}{h}) \end{aligned}$$

From $(4.1)_3$ and $(4.12)_2$

$$(4.13) \qquad \frac{\lambda}{\lambda+2\mu}\,\sigma_{\gamma\gamma,\alpha} + \sigma_{\alpha\gamma,\gamma} = \mathcal{O}(\frac{h^2}{d^3}\,\varepsilon_2) \; .$$

Substituting (4.13) into (4.10) gives

$$(4.14) \qquad \sigma_{\gamma\gamma,\nu\nu} = \frac{\lambda+2\mu}{4(\lambda+\mu)}\,(b_{\gamma\nu}b_{\gamma\nu} - b_{\gamma\gamma}b_{\nu\nu}) + \mathcal{O}(\frac{h^2}{d^4}\,\varepsilon_2) .$$

Differentiating (4.11) with respect to x_3 and setting $x_3 = 0$ gives

$$(4.15) \qquad \begin{aligned} (t_{\alpha j,j3})_0 &= \mathcal{O}(\frac{\varepsilon_1\varepsilon_2}{hd}) \\ (t_{3j,j3})_0 &= \frac{2\mu}{E}\,(b_{\gamma\nu}b_{\gamma\nu} + \frac{\lambda}{\lambda+2\mu}\,b_{\gamma\gamma}b_{\nu\nu}) + \mathcal{O}(\frac{\theta^2}{d^2}\,\varepsilon_2) = \mathcal{O}(\frac{\theta^2}{h^2}\,\varepsilon_2) \end{aligned}$$

From $(4.15)_1$, $(4.1)_4$, and $(4.9)_2$

$$(4.16) \qquad (t_{\alpha 3,33})_0 = -\frac{4\mu(\lambda+\mu)}{E(\lambda+2\mu)}\, b_{\gamma\gamma,\alpha} + \mathcal{O}(\frac{h}{d^3}\,\varepsilon_1)$$

From $(0.10)_3$ we have $(t_{3\gamma,\gamma 3})_0 = \mathcal{O}(\frac{h^2}{d^4}\,\varepsilon_2)$. This with $(4.15)_2$ gives

$$(4.17) \qquad (t_{33,33})_0 = \frac{2\mu}{E}\,(b_{\gamma\nu}b_{\gamma\nu} + \frac{\lambda}{\lambda+2\mu}\, b_{\gamma\gamma}b_{\nu\nu}) + \mathcal{O}(\frac{h^2}{d^4}\,\varepsilon_2).$$

From $(4.9)_3$ and (4.14)

$$(4.18) \qquad (e_{\gamma\gamma,33})_0 = \frac{5\lambda+4\mu}{4(\lambda+\mu)}\, b_{\gamma\nu}b_{\gamma\nu} - \frac{\lambda(5\lambda+6\mu)}{4(\lambda+\mu)(\lambda+2\mu)}\, b_{\gamma\gamma}b_{\nu\nu} + \mathcal{O}(\frac{h^2}{d^4}\,\varepsilon_2).$$

From (1.6)

$$e_{33,33} = -\frac{\lambda}{\lambda+2\mu}\, e_{\gamma\gamma,33} + \frac{E}{\lambda+2\mu}\,(t_{33,33} - z_{33,33}) .$$

From the above, (4.18), (4.17), and (4.7)

$$(4.19) \qquad (e_{33,33})_0 = L_4 b_{\gamma\nu} b_{\gamma\nu} + L_5 b_{\gamma\gamma} b_{\nu\nu} + \mathcal{O}(\frac{h^2}{d^4}\,\varepsilon_2)$$

From (4.18) and (4.19)

$$(e_{ii,33})_0 = L_{37} b_{\gamma\nu} b_{\gamma\nu} + L_{38} b_{\gamma\gamma} b_{\nu\nu} + \mathcal{O}(\frac{h^2}{d^4}\,\varepsilon_2) .$$

From (1.6), the above, $(4.9)_3$, and (4.6)

$$(4.20) \quad (t_{\alpha\beta,33})_0 = \delta_{\alpha\beta}(L_6 b_{\gamma\nu} b_{\gamma\nu} + L_7 b_{\gamma\gamma} b_{\nu\nu}) + \frac{2\lambda\mu}{E(\lambda+2\mu)}\,\sigma_{\gamma\gamma,\alpha\beta}$$
$$+ L_8 b_{\alpha\beta} b_{\gamma\gamma} + L_9 b_{\alpha\gamma} b_{\gamma\beta} + \mathcal{O}(\frac{h^2}{d^4}\,\varepsilon_2)$$

differentiating (4.11) twice with respect to x_3, letting $x_3=0$, and using $(4.9)_2$ among other previous results, we obtain

$$(t_{\alpha j,j33})_0 = \frac{8\mu(\lambda+\mu)}{E(\lambda+2\mu)} b_{\alpha\gamma} b_{\nu\nu,\gamma} - \frac{4\mu}{E} (b_{\gamma\nu} b_{\gamma\nu,\alpha}$$

(4.21)
$$+ \frac{\lambda}{\lambda+2\mu} b_{\gamma\gamma} b_{\nu\nu,\alpha}) + \mathcal{O}(\frac{\theta^2}{d^3} \varepsilon_2)$$

$$(t_{3j,j33})_0 = \mathcal{O}(\frac{\varepsilon_1 \varepsilon_2}{hd^2})$$

From (4.20), (4.14), and $(4.9)_2$

$$(t_{\alpha\gamma,\gamma 33})_0 = [\frac{4\mu(3\lambda+\mu)}{E(\lambda+ 2\mu)} + \frac{8\mu W_{20}}{\lambda+ 2\mu} + 6W_{10}] b_{\gamma\nu} b_{\gamma\nu,\alpha}$$

$$+ [- \frac{4\lambda\mu}{E(\lambda+2\mu)} + \frac{32\mu(\lambda^2+\lambda\mu+\mu^2)W_{20}+96\mu^3 W_{40}-12\lambda^3 W_{10}}{(\lambda+2\mu)^3}] b_{\gamma\gamma} b_{\nu\nu,\alpha}$$

$$+ [\frac{4\mu^2}{E(\lambda+2\mu)} + \frac{8\mu W_{20}}{\lambda+2\mu} + 6W_{10}] b_{\alpha\gamma} b_{\nu\nu,\gamma} + \mathcal{O}(\frac{h^2}{d^5} \varepsilon_2) .$$

Using this in $(4.21)_1$, we obtain

(4.22) $$(t_{\alpha 3,333})_0 = L_{10} b_{\gamma\nu} b_{\gamma\nu,\alpha} + L_{11} b_{\gamma\gamma} b_{\nu\nu,\alpha} + L_{12} b_{\alpha\gamma} b_{\nu\nu,\gamma} + \mathcal{O}(\frac{h^2}{d^5} \varepsilon_2)$$

From (4.16)

$$(t_{3\gamma,\gamma 33})_0 = - \frac{4\mu(\lambda+\mu)}{E(\lambda+2\mu)} b_{\gamma\gamma,\nu\nu} + \mathcal{O}(\frac{h}{d^4} \varepsilon_1)$$

Using this in $(4.21)_2$ gives

(4.23) $$(t_{33,333})_0 = \frac{4\mu(\lambda+\mu)}{E(\lambda+2\mu)} b_{\gamma\gamma,\nu\nu} + \mathcal{O}(\frac{h}{d^4} \varepsilon_1)$$

Using Taylor's formula, $(1.9)_2$, and the fact that $t_{\alpha 31}$ is an even function of x_3, we obtain

$$0 = t_{\alpha 31}\big|_{x_3=h} = (t_{\alpha 31})_0 + \frac{h^2}{2}(t_{\alpha 31,33})_0 + \mathcal{O}\left(\frac{h^3}{d^3}\varepsilon_1\right) .$$

This can be rewritten as

$$(t_{\alpha 3})_0 = -\frac{h^2}{2}(t_{\alpha 3,33})_0 + \mathcal{O}\left(\frac{h^3}{d^3}\varepsilon_1\right) .$$

From (4.16) this becomes

$$(t_{\alpha 3})_0 = \frac{2\mu(\lambda+\mu)}{E(\lambda+2\mu)} h^2 b_{\gamma\gamma,\alpha} + \mathcal{O}\left(\frac{h^3}{d^3}\varepsilon_1\right) . \tag{4.24}$$

From (4.24)

$$(t_{3\gamma,\gamma})_0 = \frac{2\mu(\lambda+\mu)}{E(\lambda+2\mu)} h^2 b_{\gamma\gamma,\nu\nu} + \mathcal{O}\left(\frac{h^3}{d^4}\varepsilon_1\right) .$$

Substituting this into $(4.12)_3$ gives

$$(t_{33,3})_0 = -\frac{2\mu(\lambda+\mu)}{E(\lambda+2\mu)} h^2 b_{\gamma\gamma,\nu\nu} + \frac{2\mu}{E}\left(\sigma_{\gamma\nu}b_{\gamma\nu} + \frac{\lambda}{\lambda+2\mu}\sigma_{\gamma\gamma}b_{\nu\nu}\right) + \mathcal{O}\left(\frac{h^3}{d^4}\varepsilon_1\right). \tag{4.25}$$

Also

$$0 = t_{331}\big|_{x_3=h} = h(t_{331,3})_0 + \frac{h^3}{6}(t_{331,333})_0 + \mathcal{O}\left(\frac{h^4}{d^4}\varepsilon_1\right)$$

which can be written as

$$(t_{33,3})_0 = -\frac{h^2}{6}(t_{33,333})_0 + \mathcal{O}\left(\frac{h^3}{d^4}\varepsilon_1\right) .$$

Using (4.23)

$$(t_{33,3})_0 = -\frac{2\mu(\lambda+\mu)}{3E(\lambda+2\mu)} h^2 b_{\gamma\gamma,\nu\nu} + \mathcal{O}\left(\frac{h^3}{d^4}\varepsilon_1\right). \tag{4.26}$$

From (4.25) and (4.26)

$$b_{\gamma\gamma,\nu\nu} = \frac{3(\lambda+2\mu)}{2(\lambda+\mu)}\frac{1}{h^2}\left(\sigma_{\gamma\nu}b_{\gamma\nu} + \frac{\lambda}{\lambda+2\mu}\sigma_{\gamma\gamma}b_{\nu\nu}\right) + \mathcal{O}\left(\frac{h}{d^4}\varepsilon_1\right) . \tag{4.27}$$

Both (4.26) and (4.23) can be rewritten with the aid of (4.27) as

$$(4.28)\quad \begin{aligned} (t_{33,3})_0 &= -\frac{\mu}{E}\left(\sigma_{\gamma\nu}b_{\gamma\nu} + \frac{\lambda}{\lambda+2\mu}\sigma_{\gamma\gamma}b_{\nu\nu}\right) + \mathcal{O}\left(\frac{h^3}{d^4}\varepsilon_1\right) \\ (t_{33,333})_0 &= \frac{6\mu}{E}\frac{1}{h^2}\left(\sigma_{\gamma\nu}b_{\gamma\nu} + \frac{\lambda}{\lambda+2\mu}\sigma_{\gamma\gamma}b_{\nu\nu}\right) + \mathcal{O}\left(\frac{h}{d^4}\varepsilon_1\right) . \end{aligned}$$

Also

$$0 = t_{\alpha 32}\big|_{x_3=h} = h(t_{\alpha 32,3})_0 + \frac{h^3}{6}(t_{\alpha 32,333})_0 + \mathcal{O}\left(\frac{h^5}{d^5}\varepsilon_2\right)$$

$$0 = t_{332}\big|_{x_3=h} = (t_{332})_0 + \frac{h^2}{2}(t_{332,33})_0 + \mathcal{O}\left(\frac{h^4}{d^4}\varepsilon_2\right)$$

from which we obtain (using (4.17))

$$\begin{aligned} (t_{\alpha 3,3})_0 &= -\frac{h^2}{6}(t_{\alpha 3,333})_0 + \mathcal{O}\left(\frac{h^4}{d^5}\varepsilon_2\right) \\ (4.29)\quad (t_{33})_0 &= -\frac{h^2}{2}(t_{33,33})_0 + \mathcal{O}\left(\frac{h^4}{d^4}\varepsilon_2\right) \\ &= -\frac{\mu}{E}h^2\left(b_{\gamma\nu}b_{\gamma\nu} + \frac{\lambda}{\lambda+2\mu}b_{\gamma\gamma}b_{\nu\nu}\right) + \mathcal{O}\left(\frac{h^4}{d^4}\varepsilon_2\right) . \end{aligned}$$

From $(1.6)_1$, $(4.29)_2$, and $(4.7)_1$

$$(4.30)\quad \begin{aligned} (e_{33})_0 = &-\frac{\lambda}{\lambda+2\mu}\sigma_{\gamma\gamma} - \frac{EW_{20}}{\lambda+2\mu}\sigma_{\gamma\nu}\sigma_{\gamma\nu} - \frac{EL_3}{\lambda+2\mu}\sigma_{\gamma\gamma}\sigma_{\nu\nu} \\ &- \frac{\mu}{\lambda+2\mu}h^2\left(b_{\gamma\nu}b_{\gamma\nu} + \frac{\lambda}{\lambda+2\mu}b_{\gamma\gamma}b_{\nu\nu}\right) + \mathcal{O}\left(\frac{h^4}{d^4}\varepsilon_2\right) . \end{aligned}$$

Thus

$$\begin{aligned} (e_{ii})_0 = &\frac{2\mu}{\lambda+2\mu}\sigma_{\gamma\gamma} - \frac{EW_{20}}{\lambda+2\mu}\sigma_{\gamma\nu}\sigma_{\gamma\nu} - \frac{EL_3}{\lambda+2\mu}\sigma_{\gamma\gamma}\sigma_{\nu\nu} \\ &- \frac{\mu}{\lambda+2\mu}h^2\left(b_{\gamma\nu}b_{\gamma\nu} + \frac{\lambda}{\lambda+2\mu}b_{\gamma\gamma}b_{\nu\nu}\right) + \mathcal{O}\left(\frac{h^4}{d^4}\varepsilon_2\right) . \end{aligned}$$

From $(1.6)_1$, the above, and $(4.6)_1$

$$\begin{aligned}(t_{\alpha\beta})_0 = \delta_{\alpha\beta}\{&\frac{2\lambda\mu}{E(\lambda+2\mu)}\sigma_{\gamma\gamma} + \frac{2\mu W_{20}}{\lambda+2\mu}\sigma_{\gamma\nu}\sigma_{\gamma\nu} + L_{13}\sigma_{\gamma\gamma}\sigma_{\nu\nu} \\ &- \frac{\lambda\mu}{E(\lambda+2\mu)}h^2(b_{\gamma\nu}b_{\gamma\nu} + \frac{\lambda}{\lambda+2\mu}b_{\gamma\gamma}b_{\nu\nu})\} \\ &+ \frac{2\mu}{E}\sigma_{\alpha\beta} + L_2\sigma_{\alpha\beta}\sigma_{\gamma\gamma} + 3W_{10}\sigma_{\alpha\gamma}\sigma_{\gamma\beta} + \mathcal{O}(\frac{h^4}{d^4}\varepsilon_2) .\end{aligned} \tag{4.31}$$

From $(1.6)_1$, $(4.28)_1$, $(4.7)_2$, and (0.11)

$$(e_{33,3})_0 = -\frac{\lambda}{\lambda+2\mu}b_{\gamma\gamma} + L_{14}\sigma_{\gamma\nu}b_{\gamma\nu} + L_{15}\sigma_{\gamma\gamma}b_{\nu\nu} + \mathcal{O}(\frac{h^3}{d^4}\varepsilon_1) . \tag{4.32}$$

Thus

$$(e_{ii,3})_0 = \frac{2\mu}{\lambda+2\mu}b_{\gamma\gamma} + L_{14}\sigma_{\gamma\nu}b_{\gamma\nu} + L_{15}\sigma_{\gamma\gamma}b_{\nu\nu} + \mathcal{O}(\frac{h^3}{d^4}\varepsilon_1)$$

and from $(1.6)_1$, the above, and $(4.6)_2$

$$\begin{aligned}(t_{\alpha\beta,3})_0 = \delta_{\alpha\beta}&[\frac{2\lambda\mu}{E(\lambda+2\mu)}b_{\gamma\gamma} + \frac{1}{2}L_8\sigma_{\gamma\nu}b_{\gamma\nu} + L_{16}\sigma_{\gamma\gamma}b_{\nu\nu}] \\ &+ \frac{2\mu}{E}b_{\alpha\beta} + L_2(\sigma_{\alpha\beta}b_{\gamma\gamma}+\sigma_{\gamma\gamma}b_{\alpha\beta})+3W_{10}(\sigma_{\alpha\gamma}b_{\gamma\beta}+\sigma_{\beta\gamma}b_{\gamma\alpha}) \\ &+ \mathcal{O}(\frac{h^3}{d^4}\varepsilon_1)\end{aligned} \tag{4.33}$$

From $(1.6)_2$ we have

$$z_{\alpha 3} = 2W_{20}e_{\alpha 3}s_1 + 3W_{10}e_{\alpha k}e_{k3} + e_{\alpha 3}A_2 + e_{\alpha k}e_{k3}A_1 \tag{4.34}$$

where A_i $(i=1,2)$ are analytic functions of s_1, s_2, s_3 which are of the ith order in the strains. Thus

$$(z_{\alpha 3,33})_0 = \mathcal{O}(\frac{h}{d^3}\varepsilon_1)$$

and

$$(e_{\alpha 3,33})_0 = -\frac{2(\lambda+\mu)}{\lambda+2\mu}\, b_{\gamma\gamma,\alpha} + \mathcal{O}(\frac{h}{d^3}\,\varepsilon_1) \tag{4.35}$$

using $(1.6)_1$ and (4.16).

Since $e_{\alpha 3\omega} = 0$ for $x_3 = \pm h$,

$$0 = e_{\alpha 31}|_{x_3=h} = (e_{\alpha 31})_0 + \frac{h^2}{2}(e_{\alpha 31,33})_0 + \mathcal{O}(\frac{h^3}{d^3}\,\varepsilon_1)$$

so that

$$(e_{\alpha 3})_0 = -\frac{h^2}{2}(e_{\alpha 3,33})_0 + \mathcal{O}(\frac{h^3}{d^3}\,\varepsilon_1)$$

and with (4.35)

$$(e_{\alpha 3})_0 = \frac{\lambda+\mu}{\lambda+2\mu}\, h^2 b_{\gamma\gamma,\alpha} + \mathcal{O}(\frac{h^3}{d^3}\,\varepsilon_1) \tag{4.36}$$

From (4.34)

$$(z_{\alpha 3,333})_0 = \frac{6(\lambda+\mu)(-4\mu W_{20}+3\lambda W_{10})}{(\lambda+2\mu)^2}\, b_{\gamma\gamma} b_{\nu\nu,\alpha}$$

$$-\frac{18(\lambda+\mu)W_{10}}{\lambda+2\mu}\, b_{\alpha\gamma} b_{\nu\nu,\gamma} + \mathcal{O}(\frac{\theta^2}{d^3}\,\varepsilon_2)\ .$$

From $(1.6)_1$, (4.22), and the above we have

$$\begin{aligned}(e_{\alpha 3,333})_0 = \frac{E}{2\mu}\ & L_{10} b_{\gamma\nu} b_{\gamma\nu,\alpha} + L_{17} b_{\gamma\gamma} b_{\nu\nu,\alpha} + L_{18} b_{\alpha\gamma} b_{\nu\nu,\gamma} \\ & + \mathcal{O}(\frac{h^2}{d^5}\ \varepsilon_2)\end{aligned} \tag{4.37}$$

Since

$$0 = e_{\alpha 32}|_{x_3=h} = h(e_{\alpha 32,3})_0 + \frac{h^3}{6}(e_{\alpha 32,333})_0 + \mathcal{O}(\frac{h^5}{d^5}\,\varepsilon_2)$$

we have

$$(e_{\alpha 3,3})_0 = -\frac{h^2}{6}(e_{\alpha 3,333})_0 + \mathcal{O}(\frac{h^4}{d^5}\,\varepsilon_2)$$

and from (4.37)

$$(4.38)\qquad \begin{aligned}(e_{\alpha 3,3})_0 = -\frac{h^2}{6}\Big(&\frac{E}{2\mu} L_{10} b_{\gamma\nu} b_{\gamma\nu,\alpha} + L_{17} b_{\gamma\gamma} b_{\nu\nu,\alpha}\\ &+ L_{18} b_{\alpha\gamma} b_{\nu\nu,\gamma}\Big) + \mathcal{O}\Big(\frac{h^4}{d^5}\,\varepsilon_2\Big).\end{aligned}$$

From (1.5)

$$(e_{\beta 3,\alpha 33})_0 + (e_{\alpha 3,\beta 33})_0 - (e_{\alpha\beta,333})_0 - (e_{33,\alpha\beta 3})_0 = (c_{\alpha\beta 33,3})_0 \,.$$

Using the above, (4.35), and $(4.1)_2$, this becomes

$$(e_{\alpha\beta,333})_0 = -\frac{3\lambda+4\mu}{\lambda+2\mu}\, b_{\gamma\gamma,\alpha\beta} - (c_{\alpha\beta 33,3})_0 + \mathcal{O}\Big(\frac{h}{d^4}\,\varepsilon_1\Big) \,.$$

From $(1.4)_2$ we have

$$g^{ij} = \delta_{ij} - 2e_{ij} + 4e_{ik}e_{kj} + \dots \,.$$

This with (0.11) gives

$$\left[\frac{\partial^k (g^{ij} - \delta_{ij})}{\partial x_1^{k_1}\,\partial x_2^{k_2}\,\partial x_3^{k_3}}\right]_0 = \begin{cases} \mathcal{O}\Big(\dfrac{\varepsilon_2}{d^k}\Big) & \begin{cases}\text{if } (i,j) = (\alpha,\beta) \text{ or } (3,3) \text{ and } k_3 \text{ is even}\\[4pt] \text{if } (i,j) = (\alpha,3) \text{ and } k_3 \text{ is odd}\end{cases}\\[16pt] \mathcal{O}\Big(\dfrac{\varepsilon_1}{hd^{k-1}}\Big) & \begin{cases}\text{if } (i,j) = (\alpha,\beta) \text{ or } (3,3) \text{ and } k_3 \text{ is odd}\\[4pt] \text{if } (i,j) = (\alpha,3) \text{ and } k_3 \text{ is even}\end{cases}\end{cases}$$

Then $(1.4)_3$ and $(1.4)_1$ give

$$(c_{\alpha\beta 33,3})_0 = \mathcal{O}\Big(\frac{\varepsilon_1\varepsilon_2}{hd^2}\Big)$$

so the above becomes

$$(4.39)\qquad (e_{\alpha\beta,333})_0 = -\frac{3\lambda+4\mu}{\lambda+2\mu}\, b_{\gamma\gamma,\alpha\beta} + \mathcal{O}\Big(\frac{h}{d^4}\,\varepsilon_1\Big) \,.$$

Thus

$$(e_{\gamma\gamma,333})_0 = - \frac{3\lambda+4\mu}{\lambda+2\mu} b_{\gamma\gamma,\nu\nu} + \mathcal{O}(\frac{h}{d^4} \varepsilon_1)$$

$$= - \frac{3(3\lambda+4\mu)}{2(\lambda+\mu)} \frac{1}{h^2} (\sigma_{\gamma\nu} b_{\gamma\nu} + \frac{\lambda}{\lambda+2\mu} \sigma_{\gamma\gamma} b_{\nu\nu}) + \mathcal{O}(\frac{h}{d^4} \varepsilon_1)$$

making use of (4.27).

Using $(1.6)_1$, $(4.28)_2$, $(4.7)_4$, and the above, we have

$$(e_{33,333})_0 = \frac{3(3\lambda+2\mu)}{2(\lambda+\mu)} \frac{1}{h^2} (\sigma_{\gamma\nu} b_{\gamma\nu} + \frac{\lambda}{\lambda+2\mu} \sigma_{\gamma\gamma} b_{\nu\nu}) + \mathcal{O}(\frac{h}{d^4} \varepsilon_1). \tag{4.40}$$

From (4.40) and the relation after (4.39)

$$(e_{ii,333})_0 = - \frac{3\mu}{\lambda+\mu} \frac{1}{h^2} (\sigma_{\gamma\nu} b_{\gamma\nu} + \frac{\lambda}{\lambda+2\mu} \sigma_{\gamma\gamma} b_{\nu\nu}) + \mathcal{O}(\frac{h}{d^4} \varepsilon_1) .$$

From $(1.6)_1$, the above, (4.39), and $(4.6)_3$

$$\begin{aligned}(t_{\alpha\beta,333})_0 = &- \frac{3\lambda\mu}{E(\lambda+\mu)} \frac{\delta_{\alpha\beta}}{h^2} (\sigma_{\gamma\nu} b_{\gamma\nu} + \frac{\lambda}{\lambda+2\mu} \sigma_{\gamma\gamma} b_{\nu\nu}) \\ &- \frac{2\mu(3\lambda+4\mu)}{E(\lambda+2\mu)} b_{\gamma\gamma,\alpha\beta} + \mathcal{O}(\frac{h}{d^4} \varepsilon_1) .\end{aligned} \tag{4.41}$$

From (1.5)

$$(e_{\beta 3,\alpha\gamma} + e_{\alpha\gamma,\beta 3} - e_{\alpha\beta,\gamma 3} - e_{\gamma 3,\alpha\beta})_0 = (c_{\alpha\beta\gamma 3})_0$$

which becomes

$$b_{\alpha\gamma,\beta} - b_{\alpha\beta,\gamma} = (c_{\alpha\beta\gamma 3})_0 + \mathcal{O}(\frac{h^3}{d^5} \varepsilon_1)$$

using (4.36).

From $(1.4)_3$ and the relations before (4.8)

$$
\begin{aligned}
(c_{\alpha\beta\gamma 3})_0 &= (\sigma_{\beta\nu,\alpha}+\sigma_{\alpha\nu,\beta}-\sigma_{\alpha\beta,\nu})\, b_{\gamma\nu} - (\sigma_{\gamma\nu,\alpha}+\sigma_{\alpha\nu,\gamma}-\sigma_{\alpha\gamma,\nu})\, b_{\beta\nu} \\
&\quad + \frac{\lambda}{\lambda+2\mu}\,(b_{\alpha\beta}\sigma_{\nu\nu,\gamma} - b_{\alpha\gamma}\sigma_{\nu\nu,\beta}) + \mathcal{O}(\frac{h^3}{d^5}\,\varepsilon_1) \; .
\end{aligned}
$$

Substituting this into the above and using (4.13)

$$
\begin{aligned}
b_{\alpha\gamma,\beta} - b_{\alpha\beta,\gamma} &= (\sigma_{\beta\nu,\alpha}+\sigma_{\alpha\nu,\beta}-\sigma_{\alpha\beta,\nu})b_{\gamma\nu} - (\sigma_{\gamma\nu,\alpha}+\sigma_{\alpha\nu,\gamma}-\sigma_{\alpha\gamma,\nu})b_{\beta\nu} \\
&\quad - b_{\alpha\beta}\sigma_{\gamma\nu,\nu} + b_{\alpha\gamma}\sigma_{\beta\nu,\nu} + \mathcal{O}(\frac{h^3}{d^5}\,\varepsilon_1) \; .
\end{aligned}
\tag{4.42}
$$

From (4.42) and (4.13) we obtain $(0.17)_1$.

In the same way from (1.5)

$$
\sigma_{\beta\nu,\alpha\gamma} + \sigma_{\alpha\gamma,\beta\nu} - \sigma_{\alpha\beta,\gamma\nu} - \sigma_{\gamma\nu,\alpha\beta} = (c_{\alpha\beta\gamma\nu})_0 \; .
$$

From $(1.4)_2$ and $(4.1)_1$

$$
(g^{33}-1)_0 = -\,2(e_{33})_0 + \mathcal{O}(\frac{h^2}{d^2}\,\varepsilon_2) = \frac{2\lambda}{\lambda+2\mu}\,\sigma_{\omega\omega} + \mathcal{O}(\frac{h^2}{d^2}\,\varepsilon_2) \; .
$$

From $(1.4)_1$ and (4.36)

$$
(\Gamma_{\alpha\beta 3})_0 = (e_{\beta 3,\alpha}+ e_{\alpha 3,\beta}- e_{\alpha\beta,3})_0 = -b_{\alpha\beta} + \frac{2(\lambda+\mu)}{\lambda+2\mu}\, h^2 b_{\omega\omega,\alpha\beta} + \mathcal{O}(\frac{h^3}{d^4}\,\varepsilon_1) \; .
$$

From $(1.4)_3$, the above results, and the relations before (4.8) we can calculate $(c_{\alpha\beta\gamma\nu})_0$ and substitute into a result above. This gives

$$
\begin{aligned}
&\sigma_{\beta\nu,\alpha\gamma} + \sigma_{\alpha\gamma,\beta\nu} - \sigma_{\alpha\beta,\gamma\nu} - \sigma_{\gamma\nu,\alpha\beta} \\
&= (\sigma_{\beta\omega,\alpha}+\sigma_{\alpha\omega,\beta}-\sigma_{\alpha\beta,\omega})(\sigma_{\nu\omega,\gamma}+\sigma_{\gamma\omega,\nu}-\sigma_{\gamma\nu,\omega}) \\
&\quad -(\sigma_{\gamma\omega,\alpha}+\sigma_{\alpha\omega,\gamma}-\sigma_{\alpha\gamma,\omega})(\sigma_{\nu\omega,\beta}+\sigma_{\beta\omega,\nu}-\sigma_{\beta\nu,\omega}) \\
&\quad +b_{\alpha\beta}b_{\gamma\nu} - b_{\alpha\gamma}b_{\beta\nu}
\end{aligned}
\tag{4.43}
$$

$$+ \frac{2(\lambda+\mu)}{\lambda+2\mu} h^2 (b_{\alpha\gamma} b_{\omega\omega,\beta\nu} + b_{\beta\nu} b_{\omega\omega,\alpha\gamma} - b_{\alpha\beta} b_{\omega\omega,\gamma\nu} - b_{\gamma\nu} b_{\omega\omega,\alpha\beta})$$

$$+ \frac{2\lambda}{\lambda+2\mu} \sigma_{\omega\omega} (b_{\alpha\beta} b_{\gamma\nu} - b_{\alpha\gamma} b_{\beta\nu}) + \mathcal{O}(\frac{h^2}{d^4} \theta^2 \varepsilon_2) .$$

From (4.11)

$$(t_{\alpha j,j})_0 + 2\sigma_{\alpha\gamma,\nu}(t_{\gamma\nu})_0 + 2b_{\alpha\gamma}(t_{\gamma 3})_0 - \frac{1}{E}(W_{,\alpha})_0 = \mathcal{O}(\frac{h^2}{d^3} \theta^2 \varepsilon_2) .$$

From (4.31) and (4.13)

$$(t_{\alpha\beta,\beta})_0 = \frac{2\lambda\mu}{E(\lambda+2\mu)} \sigma_{\gamma\gamma,\alpha} + \frac{2\mu}{E} \sigma_{\alpha\gamma,\gamma} + \frac{4\mu W_{20}}{\lambda+2\mu} \sigma_{\gamma\nu}\sigma_{\gamma\nu,\alpha}$$

$$+ (2L_{13} - \frac{\lambda L_2}{\lambda+2\mu}) \sigma_{\gamma\gamma}\sigma_{\nu\nu,\alpha} + (L_2 - \frac{3\lambda W_{10}}{\lambda+2\mu}) \sigma_{\alpha\gamma}\sigma_{\nu\nu,\gamma}$$

$$+ 3W_{10}\sigma_{\alpha\gamma,\nu}\sigma_{\gamma\nu} - \frac{2\lambda\mu}{E(\lambda+2\mu)} h^2 (b_{\gamma\nu} b_{\gamma\nu,\alpha} + \frac{\lambda}{\lambda+2\mu} b_{\gamma\gamma} b_{\nu\nu,\alpha})$$

$$+ \mathcal{O}(\frac{h^4}{d^5} \varepsilon_2) .$$

We use the expression for $(t_{\alpha 3,3})_0$ from $(4.29)_1$ and (4.22). From $(4.1)_3$ and (4.13)

$$2\sigma_{\alpha\gamma,\nu}(t_{\gamma\nu})_0 = - \frac{4\lambda^2\mu}{E(\lambda+2\mu)^2} \sigma_{\gamma\gamma}\sigma_{\nu\nu,\alpha} + \frac{4\mu}{E} \sigma_{\alpha\gamma,\nu}\sigma_{\gamma\nu} + \mathcal{O}(\frac{h^2}{d^3} \varepsilon_2^2) .$$

From (4.24)

$$2b_{\alpha\gamma}(t_{\gamma 3})_0 = \frac{4\mu(\lambda+\mu)}{E(\lambda+2\mu)} h^2 b_{\alpha\gamma} b_{\nu\nu,\gamma} + \mathcal{O}(\frac{h^2}{d^3} \varepsilon_1^2) .$$

From $(4.3)_1$

$$- \frac{1}{E}(W_{,\alpha})_0 = - \frac{2\mu}{E} (\sigma_{\gamma\nu}\sigma_{\gamma\nu,\alpha} + \frac{\lambda}{\lambda+2\mu} \sigma_{\gamma\gamma}\sigma_{\nu\nu,\alpha}) + \mathcal{O}(\frac{h^2}{d^3} \theta^2 \varepsilon_2) .$$

Combining these results we obtain $(0.17)_2$.

From (4.11) and $(4.3)_4$,

$$(t_{\alpha j,j333})_0 = \mathcal{O}(\frac{\varepsilon_1 \varepsilon_2}{hd^3}) .$$

From (4.41) and (4.27)

$$(t_{\alpha\gamma,\gamma 333})_0 = -\frac{12\mu}{E}\frac{1}{h^2}(\sigma_{\gamma\nu}b_{\gamma\nu}+\frac{\lambda}{\lambda+2\mu}\sigma_{\gamma\gamma}b_{\nu\nu})_{,\alpha} + \mathcal{O}(\frac{h}{d^5}\varepsilon_1) \; .$$

From (the above

$$(t_{\alpha 3,3333})_0 = \frac{12\mu}{E}\frac{1}{h^2}(\sigma_{\gamma\nu}b_{\gamma\nu}+\frac{\lambda}{\lambda+2\mu}\sigma_{\gamma\gamma}b_{\nu\nu})_{,\alpha} + \mathcal{O}(\frac{h}{d^5}\varepsilon_1) \; . \tag{4.44}$$

From (4.11), $(4.12)_1$, $(4.15)_1$, $(4.1)_3$, $(4.1)_4$, $(4.3)_2$, (4.13), and $(4.9)_2$

$$\begin{aligned}(t_{\alpha j,j3})_0 = -\frac{2\mu}{E}\{&\frac{\lambda}{\lambda+2\mu}\sigma_{\nu\nu}b_{\gamma\gamma,\alpha} + \sigma_{\gamma\nu}b_{\gamma\nu,\alpha} - \sigma_{\gamma\nu,\alpha}b_{\gamma\nu}\\ &- \frac{\lambda(3\lambda+2\mu)}{(\lambda+2\mu)^2}\sigma_{\gamma\gamma,\alpha}b_{\nu\nu} + 2\sigma_{\alpha\gamma,\nu}b_{\gamma\nu}\} + \mathcal{O}(\frac{h}{d^3}\varepsilon_1\varepsilon_2) \; .\end{aligned}$$

From (4.33), $(4.9)_2$, and (4.13)

$$\begin{aligned}(t_{\alpha\gamma,\gamma 3})_0 = \frac{2\mu}{E}&(\frac{\lambda}{\lambda+2\mu}b_{\gamma\gamma,\alpha} + b_{\alpha\gamma,\gamma}) + \frac{1}{2}L_8\sigma_{\gamma\nu,\alpha}b_{\gamma\nu}\\ &+ (\frac{1}{2}L_8+3W_{10})\sigma_{\gamma\nu}b_{\gamma\nu,\alpha} + (L_{16}-\frac{\lambda}{\lambda+2\mu}L_2)\sigma_{\gamma\gamma,\alpha}b_{\nu\nu}\\ &+ (L_{16}+L_2)\sigma_{\gamma\gamma}b_{\nu\nu,\alpha} + (L_2+3W_{10})\sigma_{\alpha\nu}b_{\gamma\gamma,\nu}\\ &+ (L_2-\frac{3\lambda W_{10}}{\lambda+2\mu})\sigma_{\gamma\gamma,\nu}b_{\alpha\nu} + 3W_{10}\sigma_{\alpha\gamma,\nu}b_{\gamma\nu} + \mathcal{O}(\frac{h^3}{d^5}\varepsilon_1) .\end{aligned}$$

From (4.42) and (4.13)

$$\begin{aligned}b_{\alpha\gamma,\gamma} = b_{\gamma\gamma,\alpha} &- \frac{2(2\lambda+\mu)}{\lambda+2\mu}\sigma_{\gamma\gamma,\nu}b_{\nu\alpha} - \sigma_{\gamma\nu,\alpha}b_{\gamma\nu}\\ &+ \frac{\lambda}{\lambda+2\mu}\sigma_{\gamma\gamma,\alpha}b_{\nu\nu} + \mathcal{O}(\frac{h^3}{d^5}\varepsilon_1) \; .\end{aligned} \tag{4.45}$$

Using (4.45) in the two above relations, we obtain

$$
\begin{aligned}
(t_{\alpha 3,33})_0 = &- \frac{4\mu(\lambda+\mu)}{E(\lambda+2\mu)} b_{\gamma\gamma,\alpha} + L_{26}\sigma_{\gamma\nu,\alpha}b_{\gamma\nu} + L_{27}\sigma_{\gamma\nu}b_{\gamma\nu,\alpha} \\
&+ L_{28}\sigma_{\gamma\gamma,\alpha}b_{\nu\nu} + L_{29}\sigma_{\gamma\gamma}b_{\nu\nu,\alpha} + L_{30}\sigma_{\alpha\nu}b_{\gamma\gamma,\nu} \\
&+ L_{31}\sigma_{\nu\nu,\gamma}b_{\gamma\alpha} + L_{32}\sigma_{\alpha\gamma,\nu}b_{\gamma\nu} + \mathcal{O}(\frac{h^3}{d^5}\varepsilon_1) .
\end{aligned}
\tag{4.46}
$$

Since

$$0 = t_{\alpha 31}|_{x_3=h} = (t_{\alpha 31})_0 + \frac{h^2}{2}(t_{\alpha 31,33})_0 + \frac{h^4}{24}(t_{\alpha 31,3333})_0 + \mathcal{O}(\frac{h^5}{d^5}\varepsilon_1),$$

we have

$$(t_{\alpha 3})_0 = -\frac{h^2}{2}(t_{\alpha 3,33})_0 - \frac{h^4}{24}(t_{\alpha 3,3333})_0 + \mathcal{O}(\frac{h^5}{d^5}\varepsilon_1) .$$

From the above, (4.46), and (4.44)

$$
\begin{aligned}
(t_{\alpha 3})_0 = h^2 \{&\frac{2\mu(\lambda+\mu)}{E(\lambda+2\mu)} b_{\gamma\gamma,\alpha} + L_{33}\sigma_{\gamma\nu,\alpha}b_{\gamma\nu} + L_{34}\sigma_{\gamma\nu}b_{\gamma\nu,\alpha} \\
&+ L_{35}\sigma_{\gamma\gamma,\alpha}b_{\nu\nu} + L_{36}\sigma_{\gamma\gamma}b_{\nu\nu,\alpha} - \frac{1}{2}L_{30}\sigma_{\alpha\nu}b_{\gamma\gamma,\nu} \\
&- \frac{1}{2}L_{31}\sigma_{\nu\nu,\gamma}b_{\gamma\alpha} - \frac{1}{2}L_{32}\sigma_{\alpha\gamma,\nu}b_{\gamma\nu}\} + \mathcal{O}(\frac{h^5}{d^5}\varepsilon_1) .
\end{aligned}
\tag{4.47}
$$

From (4.11)

$$(t_{3j,j3333})_0 = \mathcal{O}(\frac{h}{d^6}\varepsilon_1) .$$

From (4.44)

$$(t_{3\gamma,\gamma 3333})_0 = \frac{12\mu}{Eh^2}(\sigma_{\gamma\nu}b_{\gamma\nu} + \frac{\lambda}{\lambda+2\mu}\sigma_{\gamma\gamma}b_{\nu\nu})_{,\omega\omega} + \mathcal{O}(\frac{h}{d^6}\varepsilon_1) .$$

The above two results give

$$(t_{33,33333})_0 = -\frac{12\mu}{Eh^2}(\sigma_{\gamma\nu}b_{\gamma\nu} + \frac{\lambda}{\lambda+2\mu}\sigma_{\gamma\gamma}b_{\nu\nu})_{,\omega\omega} + \mathcal{O}(\frac{h}{d^6}\varepsilon_1). \tag{4.48}$$

From $(4.9)_1$ and (4.13),

$$\sigma_{\alpha\beta,\gamma\gamma} = \sigma_{\beta\gamma,\alpha\gamma} + \sigma_{\alpha\gamma,\beta\gamma} - \sigma_{\gamma\gamma,\alpha\beta} - b_{\alpha\beta}b_{\gamma\gamma} + b_{\alpha\gamma}b_{\beta\gamma} + \mathcal{O}(\frac{\theta^2}{d^2}\,\varepsilon_2)$$

$$= -\frac{3\lambda+2\mu}{\lambda+2\mu}\,\sigma_{\gamma\gamma,\alpha\beta} - b_{\alpha\beta}b_{\gamma\gamma} + b_{\alpha\gamma}b_{\beta\gamma} + \mathcal{O}(\frac{h^2}{d^4}\,\varepsilon_2)$$

so that

$$(4.49)\qquad \sigma_{\gamma\nu,\omega\omega}b_{\gamma\nu} = -\frac{3\lambda+2\mu}{\lambda+2\mu}\,\sigma_{\omega\omega,\gamma\nu}b_{\gamma\nu} - b_{\gamma\nu}b_{\gamma\nu}b_{\omega\omega} + b_{\gamma\omega}b_{\nu\omega}b_{\gamma\nu} + \mathcal{O}(\frac{h^3}{d^6}\,\varepsilon_1)\,.$$

From (4.48), (4.49), $(4.9)_2$, and (4.27)

$$(4.50)\qquad \begin{aligned}(t_{33,33333})_0 = &- \frac{18\lambda\mu}{E(\lambda+\mu)}\,\frac{1}{h^4}\,\sigma_{\omega\omega}(\sigma_{\gamma\nu}b_{\gamma\nu} + \frac{\lambda}{\lambda+2\mu}\,\sigma_{\gamma\gamma}b_{\nu\nu})\\ &- \frac{12\mu}{Eh^2}\,[-\frac{3\lambda+2\mu}{\lambda+2\mu}\,\sigma_{\omega\omega,\gamma\nu}b_{\gamma\nu} + \sigma_{\gamma\nu}b_{\omega\omega,\gamma\nu} + 2\sigma_{\gamma\nu,\omega}b_{\gamma\nu,\omega}\\ &+ \frac{2\lambda}{\lambda+2\mu}\,\sigma_{\gamma\gamma,\omega}b_{\nu\nu,\omega} + b_{\gamma\nu}b_{\nu\omega}b_{\omega\gamma} - \frac{3\lambda+4\mu}{4(\lambda+\mu)}\,b_{\gamma\nu}b_{\gamma\nu}b_{\omega\omega}\\ &- \frac{\lambda}{4(\lambda+\mu)}\,b_{\gamma\gamma}b_{\nu\nu}b_{\omega\omega}] + \mathcal{O}(\frac{h}{d^6}\,\varepsilon_1)\,.\end{aligned}$$

Next

$$(s_{1,33})_0 = (e_{ii,33})_0 = L_{37}b_{\gamma\nu}b_{\gamma\nu} + L_{38}b_{\gamma\gamma}b_{\nu\nu} + \mathcal{O}(\frac{h^2}{d^4}\,\varepsilon_2)$$

$$(s_{1,333})_0 = (e_{ii,333})_0 = \frac{-3\mu}{\lambda+\mu}\,\frac{1}{h^2}\,(\sigma_{\gamma\nu}b_{\gamma\nu} + \frac{\lambda}{\lambda+2\mu}\,\sigma_{\gamma\gamma}b_{\nu\nu}) + \mathcal{O}(\frac{h}{d^4}\,\varepsilon_1)$$

$$(4.51)\qquad \begin{aligned}(s_{2,333})_0 = &\frac{6\lambda}{\lambda+2\mu}\,\sigma_{\omega\omega,\gamma\nu}b_{\gamma\nu} - \frac{2(3\lambda+4\mu)}{\lambda+2\mu}\,b_{\omega\omega,\gamma\nu}\sigma_{\gamma\nu}\\ &- \frac{3\lambda(3\lambda+2\mu)}{(\lambda+\mu)(\lambda+2\mu)}\,\frac{1}{h^2}\,\sigma_{\omega\omega}(\sigma_{\gamma\nu}b_{\gamma\nu} + \frac{\lambda}{\lambda+2\mu}\,\sigma_{\gamma\gamma}b_{\nu\nu})\\ &+ 6b_{\gamma\nu}b_{\nu\omega}b_{\omega\gamma} - \frac{6\lambda}{\lambda+2\mu}\,(1+L_4)b_{\gamma\nu}b_{\gamma\nu}b_{\omega\omega}\\ &- \frac{6\lambda}{\lambda+2\mu}\,L_5b_{\gamma\gamma}b_{\nu\nu}b_{\omega\omega} + \mathcal{O}(\frac{h\varepsilon_1\varepsilon_2}{d^4})\end{aligned}$$

$$(s_{3,333})_0 = 6b_{\gamma\nu}b_{\nu\omega}b_{\omega\gamma} - \frac{6\lambda^3}{(\lambda+2\mu)^3}\,b_{\gamma\gamma}b_{\nu\nu}b_{\omega\omega} + \mathcal{O}(\frac{h\varepsilon_1\varepsilon_2}{d^4})\,.$$

$(4.51)_1$ follows from the formula after (4.19), and $(4.51)_2$ follows from the formula after (4.40).

$(4.51)_3$ follows from (4.2), (4.39), (4.40), $(4.9)_3$, and (4.19).

$(4.51)_4$ follows from (4.2).

As before we let a_k be an analytic function of s_1, s_2, s_3 which is of kth order in the strains. Then

$$a_4 = A_1 s_1 s_3 + A_2 s_2^2 + A_3 s_2 s_3 + A_4 s_3^2 + A_5 s_1^2 s_2 + A_6 s_1^4$$

$$a_{4,3} = a_3 s_{1,3} + a_2 s_{2,3} + a_1 s_{3,3}$$

where A_i $(i=1,\dots,6)$ are analytic functions of s_1, s_2, s_3. From (4.2) and (4.2A)

$$\begin{aligned}
(a_4)_0 &= \mathcal{O}(\theta^4 \varepsilon_2^2) \\
(a_{4,3})_0 &= \mathcal{O}\left(\frac{\theta^2 \varepsilon_1 \varepsilon_2^2}{h}\right) \\
(a_{4,33})_0 &= \mathcal{O}\left(\frac{\theta^4 \varepsilon_2^2}{h^2}\right) \\
(a_{4,333})_0 &= \mathcal{O}\left(\frac{\theta^2 \varepsilon_1 \varepsilon_2^2}{h^3}\right)
\end{aligned} \tag{4.52}$$

From (1.1) and (4.4)

$$\frac{1}{E} W = \frac{\lambda}{2E} s_1^2 + \frac{\mu}{E} s_2 + W_{10} s_3 + W_{20} s_1 s_2 + W_{40} s_1^3 + a_4$$

so that

$$\begin{aligned}
\frac{1}{E} W_{,333} = {} & \frac{\lambda}{E} (s_1 s_{1,333} + 3 s_{1,3} s_{1,33}) + \frac{\mu}{E} s_{2,333} + W_{10} s_{3,333} \\
& + W_{20} (s_{1,333} s_2 + 3 s_{1,33} s_{2,3} + 3 s_{1,3} s_{2,33} + s_1 s_{2,333}) \\
& + 3W_{40} (s_1^2 s_{1,333} + 6 s_1 s_{1,3} s_{1,33} + 2 s_{1,3}^3) + a_{4,333} .
\end{aligned}$$

From the above, (4.2), (4.51), and (4.52)

$$\begin{aligned}
\frac{1}{E}(W_{,333})_0 = {} & \frac{6\lambda\mu}{E(\lambda+2\mu)}\sigma_{\omega\omega,\gamma\nu}b_{\gamma\nu} - \frac{2\mu(3\lambda+4\mu)}{E(\lambda+2\mu)}b_{\omega\omega,\gamma\nu}\sigma_{\gamma\nu} \\
& - \frac{3\lambda\mu(3\lambda+4\mu)}{E(\lambda+\mu)(\lambda+2\mu)}\frac{1}{h^2}\sigma_{\omega\omega}(\sigma_{\gamma\nu}b_{\gamma\nu} + \frac{\lambda}{\lambda+2\mu}\sigma_{\gamma\gamma}b_{\nu\nu}) \\
& + 6(\frac{\mu}{E} + W_{10})b_{\gamma\nu}b_{\nu\omega}b_{\omega\gamma} \\
& + [\frac{6\lambda\mu}{E(\lambda+2\mu)}(L_{37}-1-L_4) + \frac{12\mu W_{20}}{\lambda+2\mu}]b_{\gamma\nu}b_{\gamma\nu}b_{\omega\omega} \\
& + [\frac{6\lambda\mu}{E(\lambda+2\mu)}(L_{38}-L_5) + \frac{6}{(\lambda+2\mu)^3}(-\lambda^3W_{10}+2\lambda^2\mu W_{20}+8\mu^3W_{40})]b_{\gamma\gamma}b_{\nu\nu}b_{\omega\omega} \\
& + \mathcal{O}(\frac{h\varepsilon_1\varepsilon_2}{d^4})
\end{aligned} \tag{4.53}$$

From (4.11) and other results derived in this section

$$\begin{aligned}
(t_{3j,j33})_0 = {} & \frac{6\lambda\mu}{E(\lambda+2\mu)}\sigma_{\omega\omega,\gamma\nu}b_{\gamma\nu} + \frac{2\lambda\mu}{E(\lambda+2\mu)}b_{\omega\omega,\gamma\nu}\sigma_{\gamma\nu} \\
& - \frac{8\lambda\mu(\lambda+\mu)}{E(\lambda+2\mu)^2}\sigma_{\gamma\gamma,\omega}b_{\nu\nu,\omega} + \frac{3\lambda^2\mu}{E(\lambda+\mu)(\lambda+2\mu)}\frac{1}{h^2}\sigma_{\omega\omega}(\sigma_{\gamma\nu}b_{\gamma\nu} + \frac{\lambda}{\lambda+2\mu}\sigma_{\gamma\gamma}b_{\nu\nu}) \\
& + 6(\frac{\mu}{E} + W_{10})b_{\gamma\nu}b_{\nu\omega}b_{\omega\gamma} + L_{39}b_{\gamma\nu}b_{\gamma\nu}b_{\omega\omega} \\
& + L_{40}b_{\gamma\gamma}b_{\nu\nu}b_{\omega\omega} + \mathcal{O}(\frac{h\varepsilon_1\varepsilon_2}{d^4}) .
\end{aligned}$$

From (4.46), (4.49), and other frequently used results

$$\begin{aligned}
(t_{3\gamma,\gamma33})_0 = {} & - \frac{4\mu(\lambda+\mu)}{E(\lambda+2\mu)}b_{\gamma\gamma,\nu\nu} + [-\frac{3\lambda+2\mu}{\lambda+2\mu}L_{26}+L_{31}-\frac{\lambda}{\lambda+2\mu}L_{32}]\sigma_{\omega\omega,\gamma\nu}b_{\gamma\nu} \\
& + (L_{27}+L_{30})\sigma_{\gamma\nu}b_{\omega\omega,\gamma\nu} + (L_{26}+L_{27}+L_{32})\sigma_{\gamma\nu,\omega}b_{\gamma\nu,\omega} \\
& + (L_{28}+L_{29} - \frac{\lambda}{\lambda+2\mu}L_{30}+L_{31})\sigma_{\gamma\gamma,\omega}b_{\nu\nu,\omega}
\end{aligned}$$

$$+ \frac{3(\lambda+2\mu)}{2(\lambda+\mu)} L_{29} \frac{1}{h^2} \sigma_{\omega\omega}(\sigma_{\gamma\nu}b_{\gamma\nu} + \frac{\lambda}{\lambda+2\mu} \sigma_{\gamma\gamma}b_{\nu\nu})$$

$$+ L_{26}b_{\gamma\nu}b_{\nu\omega}b_{\omega\gamma} + [-L_{26} + \frac{\lambda+2\mu}{4(\lambda+\mu)} L_{28}]b_{\gamma\nu}b_{\gamma\nu}b_{\omega\omega}$$

$$- \frac{\lambda+2\mu}{4(\lambda+\mu)} L_{28}b_{\gamma\gamma}b_{\nu\nu}b_{\omega\omega} + \mathcal{O}(\frac{h^3}{d^6} \varepsilon_1) .$$

From the two above results

$$\begin{aligned}(t_{33,333})_0 = {} & \frac{4\mu(\lambda+\mu)}{E(\lambda+2\mu)} b_{\gamma\gamma,\nu\nu} + L_{41} \sigma_{\omega\omega,\gamma\nu}b_{\gamma\nu} + L_{42}\sigma_{\gamma\nu}b_{\omega\omega,\gamma\nu} \\ & + L_{43}\sigma_{\gamma\nu,\omega}b_{\gamma\nu,\omega} + L_{44}\sigma_{\gamma\gamma,\omega}b_{\nu\nu,\omega} + L_{45}b_{\gamma\nu}b_{\nu\omega}b_{\omega\gamma} \\ & + L_{46}b_{\gamma\nu}b_{\gamma\nu}b_{\omega\omega} + L_{47}b_{\gamma\gamma}b_{\nu\nu}b_{\omega\omega} \\ & + L_{48} \frac{1}{h^2} \sigma_{\omega\omega}(\sigma_{\gamma\nu}b_{\gamma\nu} + \frac{\lambda}{\lambda+2\mu} \sigma_{\gamma\gamma}b_{\nu\nu}) + \mathcal{O}(\frac{h^3}{d^6} \varepsilon_1) .\end{aligned} \tag{4.54}$$

From after (4.30)

$$\begin{aligned}(s_1)_0 = (e_{ii})_0 = {} & \frac{2\mu}{\lambda+2\mu} \sigma_{\gamma\gamma} - \frac{EW_{20}}{\lambda+2\mu} \sigma_{\gamma\nu}\sigma_{\gamma\nu} - \frac{EL_3}{\lambda+2\mu} \sigma_{\gamma\gamma}\sigma_{\nu\nu} \\ & - \frac{\mu}{\lambda+2\mu} h^2(b_{\gamma\nu}b_{\gamma\nu} + \frac{\lambda}{\lambda+2\mu} b_{\gamma\gamma}b_{\nu\nu}) + \mathcal{O}(\frac{h^4}{d^4} \varepsilon_2) .\end{aligned}$$

From after (4.32)

$$(s_{1,3})_0 = (e_{ii,3})_0 = \frac{2\mu}{\lambda+2\mu} b_{\gamma\gamma} + L_{14}\sigma_{\gamma\nu}b_{\gamma\nu} + L_{15}\sigma_{\gamma\gamma}b_{\nu\nu} + \mathcal{O}(\frac{h^3}{d^4} \varepsilon_1).$$

From (4.30) and (4.32)

$$\begin{aligned}(s_{2,3})_0 = 2(e_{ij}e_{ij,3})_0 = {} & 2\sigma_{\gamma\nu}b_{\gamma\nu} + 2e_{33}e_{33,3} + \mathcal{O}(\frac{h^5}{d^6} \varepsilon_1) \\ = {} & 2\sigma_{\gamma\nu}b_{\gamma\nu} + \frac{2\lambda^2}{(\lambda+2\mu)^2} \sigma_{\gamma\gamma}b_{\nu\nu} - \frac{2\lambda L_{14}}{\lambda+2\mu} \sigma_{\gamma\nu}b_{\gamma\nu}\sigma_{\omega\omega} \\ & + \frac{2\lambda EW_{20}}{(\lambda+2\mu)^2} \sigma_{\gamma\nu}\sigma_{\gamma\nu}b_{\omega\omega} + \frac{2\lambda}{\lambda+2\mu} (\frac{EL_3}{\lambda+2\mu} - L_{15})\sigma_{\gamma\gamma}\sigma_{\nu\nu}b_{\omega\omega} \\ & + \frac{2\lambda\mu}{(\lambda+2\mu)^2} h^2b_{\omega\omega}(b_{\gamma\nu}b_{\gamma\nu} + \frac{\lambda}{\lambda+2\mu} b_{\gamma\gamma}b_{\nu\nu}) + \mathcal{O}(\frac{h^5}{d^6} \varepsilon_1) .\end{aligned}$$

From (4.2) and (4.1)

$$
\begin{aligned}
(s_{3,3})_0 &= 3(e_{ij}e_{jk}e_{ki,3})_0 \\
&= 3\sigma_{\gamma\nu}\sigma_{\nu\omega}b_{\omega\gamma} + (3e_{33}^2 e_{33,3})_0 + \mathcal{O}(\frac{h^5}{d^6}\varepsilon_1) \\
&= 3\sigma_{\gamma\nu}\sigma_{\nu\omega}b_{\omega\gamma} - \frac{3\lambda^3}{(\lambda+2\mu)^3}\sigma_{\gamma\gamma}\sigma_{\nu\nu}b_{\omega\omega} + \mathcal{O}(\frac{h^5}{d^6}\varepsilon_1) \ .
\end{aligned}
$$

From after (4.52), (4.2), and the above

$$
\begin{aligned}
\frac{1}{E}(W_{,3})_0 = {} & \frac{2\mu}{E}(\sigma_{\gamma\nu}b_{\gamma\nu} + \frac{\lambda}{\lambda+2\mu}\sigma_{\gamma\gamma}b_{\nu\nu}) + 3W_{10}\sigma_{\gamma\nu}\sigma_{\nu\omega}b_{\omega\gamma} \\
& + \frac{2\mu W_{20}}{\lambda+2\mu}(\sigma_{\gamma\nu}\sigma_{\gamma\nu}b_{\omega\omega} + 2\sigma_{\gamma\nu}b_{\gamma\nu}\sigma_{\omega\omega}) \\
& + \frac{-3\lambda^3W_{10}+6\lambda^2\mu W_{20}+24\mu^3W_{40}}{(\lambda+2\mu)^3}\sigma_{\gamma\gamma}\sigma_{\nu\nu}b_{\omega\omega} + \mathcal{O}(\frac{h^5}{d^6}\varepsilon_1) \ .
\end{aligned} \tag{4.55}
$$

From (4.11)

$$
\begin{aligned}
(t_{3j,j})_0 + 2(e_{33}t_{3j,j} + e_{3\gamma,\nu}t_{\gamma\nu} + e_{33,\gamma}t_{\gamma 3} + e_{33,3}t_{33} - \frac{1}{2E}W_{,3})_0 \\
= \mathcal{O}(\frac{h^5}{d^6}\varepsilon_1) \ .
\end{aligned}
$$

Using estimates already obtained, this becomes

$$
\begin{aligned}
(t_{3j,j})_0 = {} & - \frac{4\mu(\lambda+\mu)}{E(\lambda+2\mu)}h^2\sigma_{\gamma\nu}b_{\omega\omega,\gamma\nu} + \frac{4\lambda\mu(\lambda+\mu)}{E(\lambda+2\mu)^2}h^2\sigma_{\gamma\gamma,\omega}b_{\nu\nu,\omega} \\
& - \frac{2\lambda\mu}{E(\lambda+2\mu)}\sigma_{\omega\omega}(\sigma_{\gamma\nu}b_{\gamma\nu} + \frac{\lambda}{\lambda+2\mu}\sigma_{\gamma\gamma}b_{\nu\nu}) \\
& - \frac{2\lambda\mu}{E(\lambda+2\mu)}h^2 b_{\omega\omega}(b_{\gamma\nu}b_{\gamma\nu} + \frac{\lambda}{\lambda+2\mu}b_{\gamma\gamma}b_{\nu\nu}) + \frac{1}{E}(W_{,3})_0 + \mathcal{O}(\frac{h^5}{d^6}\varepsilon_1) \ .
\end{aligned}
$$

Combining the above with (4.55) we obtain

$$
\begin{aligned}
(t_{3j,j})_0 = {} & - \frac{4\mu(\lambda+\mu)}{E(\lambda+2\mu)}h^2\sigma_{\gamma\nu}b_{\omega\omega,\gamma\nu} + \frac{4\lambda\mu(\lambda+\mu)}{E(\lambda+2\mu)^2}h^2\sigma_{\gamma\gamma,\omega}b_{\nu\nu,\omega} \\
& + \frac{2\mu}{E}(\sigma_{\gamma\nu}b_{\gamma\nu} + \frac{\lambda}{\lambda+2\mu}\sigma_{\gamma\gamma}b_{\nu\nu}) + 3W_{10}\sigma_{\gamma\nu}\sigma_{\nu\omega}b_{\omega\gamma} \\
& + \frac{2\mu W_{20}}{\lambda+2\mu}\sigma_{\gamma\nu}\sigma_{\gamma\nu}b_{\omega\omega} + [\frac{4\mu W_{20}}{\lambda+2\mu} - \frac{2\lambda\mu}{E(\lambda+2\mu)}]\sigma_{\omega\omega}\sigma_{\gamma\nu}b_{\gamma\nu}
\end{aligned}
$$

$$
\begin{aligned}
&+ \Big[\frac{-2\lambda^2\mu}{E(\lambda+2\mu)^2} + \frac{-3\lambda^3 W_{10}+6\lambda^2\mu W_{20}+24\mu^3 W_{40}}{(\lambda+2\mu)^3}\Big]\sigma_{\gamma\gamma}\sigma_{\nu\nu}b_{\omega\omega} \\
&- \frac{2\lambda\mu}{E(\lambda+2\mu)} h^2 b_{\omega\omega}(b_{\gamma\nu}b_{\gamma\nu} + \frac{\lambda}{\lambda+2\mu} b_{\gamma\gamma}b_{\nu\nu}) + \mathcal{O}(\frac{h^5}{d^6}\,\varepsilon_1)
\end{aligned}
$$

From (4.47), (4.49), (4.27), (4.13), (4.14), etc.,

$$
\begin{aligned}
(t_{3\gamma,\gamma})_0 = h^2\{&\frac{2\mu(\lambda+\mu)}{E(\lambda+2\mu)}\, b_{\gamma\gamma,\nu\nu} + [-\frac{3\lambda+2\mu}{\lambda+2\mu} L_{33} - \frac{1}{2} L_{31} \\
&+ \frac{\lambda}{2(\lambda+2\mu)} L_{32}]\sigma_{\omega\omega,\gamma\nu}b_{\gamma\nu} \\
&+ (L_{34} - \frac{1}{2} L_{30})\sigma_{\gamma\nu}b_{\omega\omega,\gamma\nu} + (L_{33}+L_{34} - \frac{1}{2} L_{32})\sigma_{\gamma\nu,\omega}\, b_{\gamma\nu,\omega} \\
&+ [L_{35} + L_{36} + \frac{\lambda}{2(\lambda+2\mu)} L_{30} - \frac{1}{2} L_{31}]\sigma_{\gamma\gamma,\omega}b_{\nu\nu,\omega} \\
&+ L_{33}b_{\gamma\nu}b_{\nu\omega}b_{\omega\gamma} \\
&+ [-L_{33} + \frac{\lambda+2\mu}{4(\lambda+\mu)} L_{35}]b_{\gamma\nu}b_{\gamma\nu}b_{\omega\omega} - \frac{\lambda+2\mu}{4(\lambda+\mu)} L_{35}b_{\gamma\gamma}b_{\nu\nu}b_{\omega\omega}\} \\
&+ \frac{3(\lambda+2\mu)}{2(\lambda+\mu)} L_{36}\sigma_{\omega\omega}(\sigma_{\gamma\nu}b_{\gamma\nu} + \frac{\lambda}{\lambda+2\mu}\, \sigma_{\gamma\gamma}b_{\nu\nu}) + \mathcal{O}(\frac{h^5}{d^6}\,\varepsilon_1).
\end{aligned}
$$

Combining the previous results we have

$$
\begin{aligned}
(t_{33,3})_0 = {}&\frac{2\mu}{E}\,(\sigma_{\gamma\nu}b_{\gamma\nu} + \frac{\lambda}{\lambda+2\mu}\,\sigma_{\gamma\gamma}b_{\nu\nu}) + 3W_{10}\sigma_{\gamma\omega}\sigma_{\omega\nu}b_{\nu\gamma} \\
&+ \frac{2\mu W_{20}}{\lambda+2\mu}\,\sigma_{\gamma\nu}\sigma_{\gamma\nu}b_{\omega\omega} + L_{49}\sigma_{\omega\omega}\sigma_{\gamma\nu}b_{\gamma\nu} + L_{50}\sigma_{\gamma\gamma}\sigma_{\nu\nu}b_{\omega\omega} \\
&+ h^2\,\{-\frac{2\mu(\lambda+\mu)}{E(\lambda+2\mu)}\, b_{\gamma\gamma,\nu\nu} + L_{51}\sigma_{\omega\omega,\gamma\nu}b_{\gamma\nu} + L_{52}\sigma_{\gamma\nu}b_{\omega\omega,\gamma\nu} \\
&+ L_{53}\sigma_{\gamma\nu,\omega}b_{\gamma\nu,\omega} + L_{54}\sigma_{\gamma\gamma,\omega}b_{\nu\nu,\omega} - L_{33}b_{\gamma\nu}b_{\nu\omega}b_{\omega\gamma} \\
&+ L_{55}b_{\gamma\nu}b_{\gamma\nu}b_{\omega\omega} + L_{56}b_{\gamma\gamma}b_{\nu\nu}b_{\omega\omega}\} + \mathcal{O}(\frac{h^5}{d^6}\,\varepsilon_1)\;.
\end{aligned}
\tag{4.56}
$$

Also

$$0 = t_{331}\big|_{x_3=h} = h(t_{331,3})_0 + \frac{h^3}{6}(t_{331,333})_0 + \frac{h^5}{120}(t_{331,33333})_0 + \mathcal{O}(\frac{h^6}{d^6}\,\varepsilon_1)$$

which becomes

$$(t_{33,3})_0 = -\frac{h^2}{6}(t_{33,333})_0 - \frac{h^4}{120}(t_{33,33333})_0 + \mathcal{O}(\frac{h^5}{d^6}\,\varepsilon_1) \; .$$

Using (4.54) and (4.50) the above becomes

$$\begin{aligned}(t_{33,3})_0 &= L_{57}\sigma_{\omega\omega}(\sigma_{\gamma\nu}b_{\gamma\nu} + \frac{\lambda}{\lambda+2\mu}\sigma_{\gamma\gamma}b_{\nu\nu})\\ &+ h^2\{-\frac{2\mu(\lambda+\mu)}{3E(\lambda+2\mu)}b_{\gamma\gamma,\nu\nu} + L_{58}\sigma_{\omega\omega,\gamma\nu}b_{\gamma\nu} + L_{59}\sigma_{\gamma\nu}b_{\omega\omega,\gamma\nu}\\ &+ L_{60}\sigma_{\gamma\nu,\omega}b_{\gamma\nu,\omega} + L_{61}\sigma_{\gamma\gamma,\omega}b_{\nu\nu,\omega} + L_{62}b_{\gamma\nu}b_{\nu\omega}b_{\omega\gamma}\\ &+ L_{63}b_{\gamma\nu}b_{\gamma\nu}b_{\omega\omega} + L_{64}b_{\gamma\gamma}b_{\nu\nu}b_{\omega\omega}\} + \mathcal{O}(\frac{h^5}{d^6}\,\varepsilon_1) \; .\end{aligned} \tag{4.57}$$

From (4.56) and (4.57) we now obtain $(0.17)_4$.

From (4.43), (4.13), and (4.27)

$$\begin{aligned}&\sigma_{\gamma\gamma,\nu\nu} - \sigma_{\gamma\nu,\gamma\nu} - \frac{3}{2}\sigma_{\gamma\nu,\omega}\,\sigma_{\gamma\nu,\omega} + \sigma_{\gamma\nu,\omega}\,\sigma_{\gamma\omega,\nu} + \frac{(3\lambda+2\mu)^2}{2(\lambda+2\mu)^2}\sigma_{\gamma\gamma,\omega}\,\sigma_{\nu\nu,\omega}\\ &\quad - \frac{1}{2}b_{\gamma\nu}b_{\gamma\nu} + \frac{1}{2}b_{\gamma\gamma}b_{\nu\nu} - 3b_{\omega\omega}(\sigma_{\gamma\nu}b_{\gamma\nu} + \frac{\lambda}{\lambda+2\mu}\sigma_{\gamma\gamma}b_{\nu\nu})\\ &\quad + \frac{2(\lambda+\mu)}{\lambda+2\mu}h^2 b_{\gamma\nu}b_{\omega\omega,\gamma\nu} - \frac{\lambda}{\lambda+2\mu}\sigma_{\omega\omega}(b_{\gamma\nu}b_{\gamma\nu} - b_{\gamma\gamma}b_{\nu\nu})\\ &= \mathcal{O}(\frac{h^4}{d^6}\,\varepsilon_2) \; .\end{aligned}$$

From $(0.17)_2$, (4.43), (4.14), (4.13), $(4.9)_2$, and (4.27)

$$\begin{aligned}&\frac{\lambda}{\lambda+2\mu}\sigma_{\gamma\gamma,\nu\nu} + \sigma_{\gamma\nu,\gamma\nu} + (L_{21} - \frac{3\lambda+2\mu}{\lambda+2\mu}L_{19} - \frac{\lambda}{\lambda+2\mu}L_{22})\sigma_{\omega\omega,\gamma\nu}\,\sigma_{\gamma\nu}\\ &+ L_{19}\sigma_{\gamma\nu,\omega}\sigma_{\gamma\nu,\omega} + L_{22}\sigma_{\gamma\nu,\omega}\sigma_{\gamma\omega,\nu} + [L_{20} - \frac{\lambda}{\lambda+2\mu}L_{21}]\sigma_{\gamma\gamma,\omega}\sigma_{\nu\nu,\omega} +\end{aligned}$$

$$+ L_{19} b_{\gamma\nu} b_{\nu\omega} \sigma_{\omega\gamma} + [\frac{3(\lambda+2\mu)}{2(\lambda+\mu)} L_{24} - L_{19}] \sigma_{\gamma\nu} b_{\gamma\nu} b_{\omega\omega}$$

$$+ \frac{\lambda+2\mu}{4(\lambda+\mu)} L_{20} \sigma_{\omega\omega} b_{\gamma\nu} b_{\gamma\nu} + [\frac{3\lambda}{2(\lambda+\mu)} L_{24} - \frac{\lambda+2\mu}{4(\lambda+\mu)} L_{20}] \sigma_{\gamma\gamma} b_{\nu\nu} b_{\omega\omega}$$

$$+ h^2 \{ (L_{23}+L_{25}) b_{\omega\omega,\gamma\nu} b_{\gamma\nu} + L_{23} b_{\gamma\nu,\omega} b_{\gamma\nu,\omega} + (L_{24}+L_{25}) b_{\gamma\gamma,\omega} b_{\nu\nu,\omega} \}$$

$$= \mathcal{O}(\frac{h^4}{d^6} \varepsilon_2) .$$

Combining the above two results, we obtain $(0.17)_3$. This completes the derivation of (0.17).

(0.14) and (0.15) are now easily verified using Taylor's formula and estimates we have derived. In the listing below the left-hand column gives a part of (0.14) or (0.15), and the formula numbers to the right indicate the estimates which verify the part of (0.14) or (0.15).

$(0.14)_1$: $(0.2)_2$, (4.39)
$(0.14)_2$: $(0.2)_1$, $(4.9)_3$
$(0.14)_3$: (4.36), (4.35)
$(0.14)_4$: (4.38), (4.37)
$(0.14)_5$: (4.32), (4.40)
$(0.14)_6$: (4.30), (4.19)
$(0.15)_1$: (4.33), (4.41)
$(0.15)_2$: (4.31), (4.20)
$(0.15)_3$: (4.47), (4.46), (4.44)
$(0.15)_4$: $(4.29)_1$, (4.22)
$(0.15)_5$: (4.57), (4.54), (4.50)
$(0.15)_6$: $(4.29)_2$, (4.17).

This completes the derivation of (0.14) and (0.15).

Appendix

$$L_1 = \frac{\lambda^2 W_{20} + 12\mu^2 W_{40}}{(\lambda+ 2\mu)^2}$$

$$L_2 = \frac{4\mu W_{20}}{\lambda+2\mu}$$

$$L_3 = \frac{\lambda(\lambda-4\mu)W_{20}+12\mu^2 W_{40}+3\lambda^2 W_{10}}{(\lambda+ 2\mu)^2}$$

$$L_4 = \frac{-5\lambda^2+ 4\lambda\mu+ 8\mu^2}{4(\lambda+\mu)(\lambda+2\mu)} - \frac{2EW_{20}}{\lambda +2\mu}$$

$$L_5 = \frac{\lambda(5\lambda+4\mu)}{4(\lambda+\mu)(\lambda+2\mu)} - \frac{2EL_3}{\lambda+2\mu}$$

$$= \frac{\lambda(5\lambda+4\mu)}{4(\lambda+\mu)(\lambda+2\mu)} - 2E \frac{\lambda(\lambda-4\mu)W_{20}+12\mu^2 W_{40}+3\lambda^2 W_{10}}{(\lambda+2\mu)^3}$$

$$L_6 = \frac{\lambda(5\lambda+4\mu)}{4E(\lambda+\mu)} + \frac{\lambda L_4}{E} + 2W_{20} = \frac{\lambda\mu(9\lambda+ 8\mu)}{2E(\lambda+\mu)(\lambda+2\mu)} + \frac{4\mu W_{20}}{\lambda+2\mu}$$

$$L_7 = -\frac{\lambda^2(5\lambda+6\mu)}{4E(\lambda+\mu)(\lambda+2\mu)} + \frac{\lambda L_5}{E} + 2L_1$$

$$= \frac{-\lambda^2\mu}{2E(\lambda+\mu)(\lambda+2\mu)} + \frac{12\lambda^2\mu W_{20}+48\mu^3 W_{40}-6\lambda^3 W_{10}}{(\lambda+2\mu)^3}$$

$$L_8 = -\frac{2\lambda\mu}{E(\lambda+2\mu)} + 2L_2 = \frac{2\mu(4EW_{20}-\lambda)}{E(\lambda+2\mu)}$$

$$L_9 = \frac{2\mu}{E} + 6W_{10}$$

$$L_{10} = -\frac{\mu(5\lambda+4\mu)}{E(\lambda+\mu)} - 2L_6-L_9 = -\frac{4\mu(4\lambda+3\mu)}{E(\lambda+2\mu)} - \frac{8\mu W_{20}}{\lambda+2\mu} - 6W_{10}$$

$$L_{11} = -\frac{\lambda\mu(3\lambda+ 2\mu)}{E(\lambda+\mu)(\lambda+2\mu)} -2L_7-L_8 = -\frac{32\mu(\lambda^2+\lambda\mu+\mu^2)W_{20}+96\mu^3 W_{40}-12\lambda^3 W_{10}}{(\lambda+2\mu)^3}$$

$$L_{12} = - L_8 - L_9 + \frac{8\mu(\lambda+\mu)}{E(\lambda+2\mu)} = \frac{4\mu(2\lambda+\mu)}{E(\lambda+2\mu)} - \frac{8\mu W_{20}}{\lambda+2\mu} - 6W_{10}$$

$$L_{13} = L_1 - \frac{\lambda L_3}{\lambda+2\mu} = \frac{6\lambda^2\mu W_{20} + 24\mu^3 W_{40} - 3\lambda^3 W_{10}}{(\lambda+2\mu)^3}$$

$$L_{14} = - \frac{\mu}{\lambda+2\mu} - \frac{2EW_{20}}{\lambda+2\mu}$$

$$L_{15} = - \frac{\lambda\mu}{(\lambda+2\mu)^2} - \frac{2EL_3}{\lambda+2\mu}$$

$$L_{16} = \frac{\lambda}{E} L_{15} + 2L_1 = \frac{-\lambda^2\mu}{E(\lambda+2\mu)^2} + \frac{12\lambda^2\mu W_{20} + 48\mu^3 W_{40} - 6\lambda^3 W_{10}}{(\lambda+2\mu)^3}$$

$$L_{17} = \frac{E}{\mu} \frac{4\mu(-\lambda^2+5\lambda\mu+2\mu^2)W_{20} - 48\mu^3 W_{40} - 3\lambda(\lambda^2+9\lambda\mu+6\mu^2)W_{10}}{(\lambda+2\mu)^3}$$

$$L_{18} = \frac{2(2\lambda+\mu)}{\lambda+2\mu} - \frac{4EW_{20}}{\lambda+2\mu} + \frac{3E(2\lambda+\mu)}{\mu(\lambda+2\mu)} W_{10}$$

$$L_{19} = \frac{2EW_{20}}{\lambda+2\mu} - 1$$

$$L_{20} = \frac{E}{\mu} L_{13} - \frac{\lambda E L_2}{2\mu(\lambda+2\mu)} - \frac{\lambda(3\lambda+2\mu)}{(\lambda+2\mu)^2}$$

$$= \frac{E}{\mu} \frac{4\lambda\mu(\lambda-\mu)W_{20} + 24\mu^3 W_{40} - 3\lambda^3 W_{10}}{(\lambda+2\mu)^3} - \frac{\lambda(3\lambda+2\mu)}{(\lambda+ 2\mu)^2}$$

$$L_{21} = \frac{EL_2}{2\mu} - \frac{3\lambda E W_{10}}{2\mu(\lambda+2\mu)} = \frac{2EW_{20}}{\lambda+2\mu} - \frac{3\lambda E W_{10}}{2\mu(\lambda+2\mu)}$$

$$L_{22} = \frac{3EW_{10}}{2\mu} + 2$$

$$L_{23} = - \frac{\lambda}{\lambda+2\mu} - \frac{EL_{10}}{12\mu} = \frac{\lambda+3\mu}{3(\lambda+2\mu)} + \frac{2EW_{20}}{3(\lambda+2\mu)} + \frac{E}{2\mu} W_{10}$$

$$L_{24} = -\frac{\lambda^2}{(\lambda+2\mu)^2} - \frac{EL_{11}}{12\mu} = \frac{-\lambda^2}{(\lambda+2\mu)^2} + \frac{E}{3\mu}\,\frac{8\mu(\lambda^2+\lambda\mu+\mu^2)W_{20}+24\mu^3W_{40}-3\lambda^3W_{10}}{(\lambda+2\mu)^3}$$

$$L_{25} = -\frac{EL_{12}}{12\mu} + \frac{2(\lambda+\mu)}{\lambda+2\mu} = \frac{4\lambda+5\mu}{3(\lambda+2\mu)} + \frac{2EW_{20}}{3(\lambda+2\mu)} + \frac{E}{2\mu}W_{10}$$

$$L_{26} = \frac{4\mu}{E} - \frac{1}{2}L_8 = \frac{\mu(5\lambda+8\mu)}{E(\lambda+2\mu)} - \frac{4\mu W_{20}}{\lambda+2\mu}$$

$$L_{27} = -\frac{2\mu}{E} - \frac{1}{2}L_8 - 3W_{10} = -\frac{\mu(\lambda+4\mu)}{E(\lambda+2\mu)} - \frac{4\mu W_{20}}{\lambda+2\mu} - 3W_{10}$$

$$L_{28} = \frac{4\lambda^2\mu}{E(\lambda+2\mu)^2} - L_{16} + \frac{\lambda}{\lambda+2\mu}L_2$$

$$= \frac{5\lambda^2\mu}{E(\lambda+2\mu)^2} - \frac{8\lambda\mu(\lambda-\mu)W_{20}+48\mu^3W_{40}-6\lambda^3W_{10}}{(\lambda+2\mu)^3}$$

$$L_{29} = -\frac{2\lambda\mu}{E(\lambda+2\mu)} - L_{16} - L_2$$

$$= -\frac{\lambda\mu(\lambda+4\mu)}{E(\lambda+2\mu)^2} - \frac{16\mu(\lambda^2+\lambda\mu+\mu^2)W_{20}+48\mu^3W_{40}-6\lambda^3W_{10}}{(\lambda+2\mu)^3}$$

$$L_{30} = -L_2 - 3W_{10} = -\frac{4\mu W_{20}}{\lambda+2\mu} - 3W_{10}$$

$$L_{31} = -L_2 + \frac{3\lambda W_{10}}{\lambda+2\mu} + \frac{4\mu(2\lambda+\mu)}{E(\lambda+2\mu)} = -\frac{4\mu W_{20}}{\lambda+2\mu} + \frac{3\lambda W_{10}}{\lambda+2\mu} + \frac{4\mu(2\lambda+\mu)}{E(\lambda+2\mu)}$$

$$L_{32} = -\frac{4\mu}{E} - 3W_{10}$$

$$L_{33} = -\frac{1}{2}L_{26} - \frac{\mu}{2E} = -\frac{\mu(3\lambda+5\mu)}{E(\lambda+2\mu)} + \frac{2\mu W_{20}}{\lambda+2\mu}$$

$$L_{34} = -\frac{1}{2}L_{27} - \frac{\mu}{2E} = \frac{\mu^2}{E(\lambda+2\mu)} + \frac{2\mu W_{20}}{\lambda+2\mu} + \frac{3}{2}W_{10}$$

$$L_{35} = -\frac{1}{2} L_{28} - \frac{\lambda\mu}{2E(\lambda+2\mu)} = -\frac{\lambda\mu(3\lambda+\mu)}{E(\lambda+2\mu)^2} + \frac{4\lambda\mu(\lambda-\mu)W_{20}+24\mu^3 W_{40}-3\lambda^3 W_{10}}{(\lambda+2\mu)^3}$$

$$L_{36} = -\frac{1}{2} L_{29} - \frac{\lambda\mu}{2E(\lambda+2\mu)} = \frac{\lambda\mu^2}{E(\lambda+2\mu)^2} + \frac{8\mu(\lambda^2+\lambda\mu+\mu^2)W_{20}+24\mu^3 W_{40}-3\lambda^3 W_{10}}{(\lambda+2\mu)^3}$$

$$L_{37} = \frac{\mu(9\lambda+8\mu)}{2(\lambda+\mu)(\lambda+2\mu)} - \frac{2EW_{20}}{\lambda+2\mu}$$

$$L_{38} = -\frac{\lambda\mu}{2(\lambda+\mu)(\lambda+2\mu)} - 2E\,\frac{\lambda(\lambda-4\mu)W_{20}+12\mu^2 W_{40}+3\lambda^2 W_{10}}{(\lambda+2\mu)^3}$$

$$L_{39} = \frac{6\lambda\mu}{E(\lambda+2\mu)}(L_{37}+1-L_4) + \frac{12\mu W_{20}}{\lambda+2\mu} = \frac{3\lambda\mu(9\lambda+8\mu)}{2E(\lambda+\mu)(\lambda+2\mu)} + \frac{12\mu W_{20}}{\lambda+2\mu}$$

$$L_{40} = \frac{6\lambda\mu}{E(\lambda+2\mu)}\left(L_{38}-L_5+\frac{2\lambda}{\lambda+2\mu}\right) + \frac{6}{(\lambda+2\mu)^3}(-\lambda^3 W_{10}+2\lambda^2\mu W_{20}+8\mu^3 W_{40})$$

$$= \frac{3\lambda^2\mu(3\lambda+2\mu)}{2E(\lambda+\mu)(\lambda+2\mu)^2} + \frac{6}{(\lambda+2\mu)^3}(-\lambda^3 W_{10} + 2\lambda^2\mu W_{20} + 8\mu^3 W_{40})$$

$$L_{41} = \frac{6\lambda\mu}{E(\lambda+2\mu)} + \frac{3\lambda+2\mu}{\lambda+2\mu} L_{26} - L_{31} + \frac{\lambda}{\lambda+2\mu} L_{32}$$

$$= \frac{\mu(3\lambda+2\mu)(3\lambda+4\mu)}{E(\lambda+2\mu)^2} - 2\lambda\left[\frac{4\mu W_{20}}{(\lambda+2\mu)^2} + \frac{3W_{10}}{\lambda+2\mu}\right]$$

$$L_{42} = \frac{2\lambda\mu}{E(\lambda+2\mu)} - L_{27} - L_{30} = \frac{\mu(3\lambda+4\mu)}{E(\lambda+2\mu)} + \frac{8\mu W_{20}}{\lambda+2\mu} + 6W_{10}$$

$$L_{43} = -L_{26}-L_{27}-L_{32} = \frac{4\mu^2}{E(\lambda+2\mu)} + \frac{8\mu W_{20}}{\lambda+2\mu} + 6W_{10}$$

$$L_{44} = -\frac{8\lambda\mu(\lambda+\mu)}{E(\lambda+2\mu)^2} - L_{28} - L_{29} + \frac{\lambda}{\lambda+2\mu} L_{30} - L_{31}$$

$$= -\frac{4\mu(5\lambda^2+6\lambda\mu+2\mu^2)}{E(\lambda+2\mu)^2}$$

$$+ \frac{8\mu(3\lambda^2+2\lambda\mu+4\mu^2)W_{20}+96\mu^3 W_{40}-6\lambda(3\lambda^2+4\lambda\mu+4\mu^2)W_{10}}{(\lambda+2\mu)^3}$$

$$L_{45} = \frac{6\mu}{E} + 6W_{10} - L_{26} = \frac{\mu(\lambda+4\mu)}{E(\lambda+2\mu)} + \frac{4\mu W_{20}}{\lambda+2\mu} + 6W_{10}$$

$$L_{46} = L_{39} + L_{26} - \frac{\lambda+2\mu}{4(\lambda+\mu)} L_{28}$$

$$= \frac{\mu(69\lambda^2+100\lambda\mu+32\mu^2)}{4E(\lambda+\mu)(\lambda+2\mu)} + \frac{4\mu(5\lambda^2+11\lambda\mu+8\mu^2)W_{20}+ 24\mu^3W_{40}- 3\lambda^3W_{10}}{2(\lambda+\mu)(\lambda+2\mu)^2}$$

$$L_{47} = L_{40} + \frac{\lambda+2\mu}{4(\lambda+\mu)} L_{28} = \frac{\lambda^2\mu(23\lambda+22\mu)}{4E(\lambda+\mu)(\lambda+2\mu)^2} - \frac{4\lambda\mu(\lambda-\mu)W_{20}+24\mu^3W_{40}-3\lambda^3W_{10}}{2(\lambda+\mu)(\lambda+2\mu)^2}$$

$$L_{48} = \frac{3\lambda^2\mu}{E(\lambda+2\mu)(\lambda+\mu)} - \frac{3(\lambda+2\mu)}{2(\lambda+\mu)} L_{29}$$

$$= \frac{3\lambda\mu(3\lambda+4\mu)}{2E(\lambda+\mu)(\lambda+2\mu)} + 3\,\frac{8\mu(\lambda^2+\lambda\mu+\mu^2)W_{20}+ 24\mu^3W_{40}- 3\lambda^3W_{10}}{(\lambda+\mu)\ (\lambda+2\mu)^2}$$

$$L_{49} = \frac{4\mu W_{20}}{\lambda+2\mu} - \frac{2\lambda\mu}{E(\lambda+2\mu)} - \frac{3(\lambda+2\mu)}{2(\lambda+\mu)}\ L_{36}$$

$$= \frac{-\lambda\mu(4\lambda+ 7\mu)}{2E(\lambda+\mu)(\lambda+2\mu)} - \frac{8\mu(2\lambda^2+\mu^2)W_{20}+ 72\mu^3W_{40}- 9\lambda^3W_{10}}{2(\lambda+\mu)(\lambda+2\mu)^2}$$

$$L_{50} = - \frac{2\lambda^2\mu}{E(\lambda+2\mu)^2} + 3\,\frac{-\lambda^3W_{10}+ 2\lambda^2\mu W_{20}+ 8\mu^3W_{40}}{(\lambda+2\mu)^3} - \frac{3\lambda}{2(\lambda+\mu)} L_{36}$$

$$= \frac{-\lambda^2\mu(4\lambda+ 7\mu)}{2E(\lambda+\mu)(\lambda+2\mu)^2} + \frac{-12\lambda\mu(\lambda^2+\lambda\mu+2\mu^2)W_{20}+24\mu^3(-\lambda+2\mu)W_{40}+3\lambda^3(\lambda-2\mu)W_{10}}{2(\lambda+\mu)(\lambda+2\mu)^3}$$

$$L_{51} = \frac{3\lambda+2\mu}{\lambda+2\mu} L_{33} + \frac{1}{2} L_{31} - \frac{\lambda}{2(\lambda+2\mu)} L_{32}$$

$$= - \frac{\mu(3\lambda^2+7\lambda\mu+6\mu^2)}{E(\lambda+2\mu)^2} + \frac{4\lambda\mu W_{20}}{(\lambda+2\mu)^2} + \frac{3\lambda W_{10}}{\lambda+2\mu}$$

$$L_{52} = - L_{34} + \frac{1}{2} L_{30} - \frac{4\mu(\lambda+\mu)}{E(\lambda+2\mu)} = \frac{-\mu(4\lambda+5\mu)}{E(\lambda+2\mu)} - \frac{4\mu W_{20}}{\lambda+2\mu} - 3\ W_{10}$$

$$L_{53} = - L_{33} - L_{34} + \frac{1}{2} L_{32} = \frac{\lambda\mu}{E(\lambda+2\mu)} - \frac{4\mu W_{20}}{\lambda+2\mu} - 3W_{10}$$

$$L_{54} = - L_{35} - L_{36} - \frac{\lambda}{2(\lambda+2\mu)} L_{30} + \frac{1}{2} L_{31} + \frac{4\lambda\mu(\lambda+\mu)}{E(\lambda+2\mu)^2}$$

$$= \frac{\mu(11\lambda^2+14\lambda\mu+4\mu^2)}{E(\lambda+2\mu)^2} - \frac{4\mu(3\lambda^2+2\lambda\mu+4\mu^2)W_{20}+48\mu^3 W_{40}-3\lambda(3\lambda^2+4\lambda\mu+4\mu^2)W_{10}}{(\lambda+2\mu)^3}$$

$$L_{55} = L_{33} - \frac{\lambda+2\mu}{4(\lambda+\mu)} L_{35} - \frac{2\lambda\mu}{E(\lambda+2\mu)}$$

$$= \frac{-\mu(17\lambda^2+39\lambda\mu+20\mu^2)}{4E(\lambda+\mu)(\lambda+2\mu)} + \frac{4\mu(\lambda^2+7\lambda\mu+4\mu^2)W_{20}-24\mu^3 W_{40}+3\lambda^3 W_{10}}{4(\lambda+\mu)(\lambda+2\mu)^2}$$

$$L_{56} = \frac{\lambda+2\mu}{4(\lambda+\mu)} L_{35} - \frac{2\lambda^2\mu}{E(\lambda+2\mu)^2}$$

$$= \frac{-\lambda\mu(11\lambda^2+15\lambda\mu+2\mu^2)}{4E(\lambda+\mu)(\lambda+2\mu)^2} + \frac{4\lambda\mu(\lambda-\mu)W_{20}+ 24\mu^3 W_{40}- 3\lambda^3 W_{10}}{4(\lambda+\mu)(\lambda+2\mu)^2}$$

$$L_{57} = \frac{3\lambda\mu}{20E(\lambda+\mu)} - \frac{1}{6} L_{48}$$

$$= \frac{- \lambda\mu(6\lambda+ 7\mu)}{10E(\lambda+\mu)(\lambda+2\mu)} - \frac{8\mu(\lambda^2+\lambda\mu+\mu^2)W_{20}+ 24\mu^3 W_{40}- 3\lambda^3 W_{10}}{2(\lambda+\mu)(\lambda+2\mu)^2}$$

$$L_{58} = - \frac{1}{6} L_{41} - \frac{\mu(3\lambda+2\mu)}{10E(\lambda+2\mu)} = \frac{-\mu(3\lambda+2\mu)(9\lambda+13\mu)}{15E(\lambda+2\mu)^2} + \frac{\lambda}{3(\lambda+2\mu)}\left(\frac{4\mu W_{20}}{\lambda+2\mu} +3W_{10}\right)$$

$$L_{59} = - \frac{1}{6} L_{42} + \frac{\mu}{10E} = \frac{-\mu(6\lambda+7\mu)}{15E(\lambda+2\mu)} - \frac{4\mu W_{20}}{3(\lambda+2\mu)} - W_{10}$$

$$L_{60} = - \frac{1}{6} L_{43} + \frac{\mu}{5E} = \frac{\mu(3\lambda- 4\mu)}{15E(\lambda+2\mu)} - \frac{4\mu W_{20}}{3(\lambda+2\mu)} - W_{10}$$

$$L_{61} = - \frac{1}{6} L_{44}+ \frac{\lambda\mu}{5E(\lambda+2\mu)} = \frac{\mu(53\lambda^2+66\lambda\mu+20\mu^2)}{15E(\lambda+2\mu)^2}$$

$$- \frac{4\mu(3\lambda^2+2\lambda\mu+4\mu^2)W_{20}+ 48\mu^3 W_{40}- 3\lambda(3\lambda^2+4\lambda\mu+4\mu^2)W_{10}}{3(\lambda+2\mu)^3}$$

$$L_{62} = -\frac{1}{6} L_{45} + \frac{\mu}{10E} = \frac{-\mu(\lambda+7\mu)}{15E(\lambda+2\mu)} - \frac{2\mu W_{20}}{3(\lambda+2\mu)} - W_{10}$$

$$L_{63} = -\frac{1}{6} L_{46} - \frac{\mu(3\lambda+4\mu)}{40E(\lambda+\mu)}$$

$$= \frac{-\mu(177\lambda^2+265\lambda\mu+92\mu^2)}{60E(\lambda+\mu)(\lambda+2\mu)} - \frac{4\mu(5\lambda^2+11\lambda\mu+8\mu^2)W_{20} + 24\mu^3 W_{40} - 3\lambda^2 W_{10}}{12(\lambda+\mu)(\lambda+2\mu)^2}$$

$$L_{64} = -\frac{1}{6} L_{47} - \frac{\lambda\mu}{40E(\lambda+\mu)}$$

$$= \frac{-\lambda\mu(59\lambda^2+61\lambda\mu+6\mu^2)}{60E(\lambda+2\mu)^2(\lambda+\mu)}$$

$$- \frac{4\lambda\mu(5\lambda^2+5\lambda\mu+2\mu^2)W_{20} + 24\mu^3(3\lambda+2\mu)W_{40} - 3\lambda^3(3\lambda+2\mu)W_{10}}{12(\lambda+\mu)(\lambda+2\mu)^3}$$

$$L_{65} = \frac{9E(\lambda+2\mu)}{4\mu(\lambda+\mu)} W_{10}$$

$$L_{66} = \frac{3EW_{20}}{2(\lambda+\mu)}$$

$$L_{67} = \frac{3E(\lambda+2\mu)}{4\mu(\lambda+\mu)} (L_{49}-L_{57})$$

$$= \frac{-21\lambda(\lambda+2\mu)}{20(\lambda+\mu)^2} - 3E \frac{4\lambda\mu(\lambda-\mu)W_{20} + 24\mu^3 W_{40} - 3\lambda^3 W_{10}}{4\mu(\lambda+\mu)^2(\lambda+2\mu)}$$

$$L_{68} = \frac{3E(\lambda+2\mu)}{4\mu(\lambda+\mu)} \left(L_{50} - \frac{\lambda}{\lambda+2\mu} L_{57}\right)$$

$$= \frac{-21\lambda^2}{20(\lambda+\mu)^2} + 3E \frac{-2\lambda(\lambda^2+\lambda\mu+4\mu^2)W_{20} + 24\mu^3 W_{40} - 3\lambda^3 W_{10}}{4(\lambda+\mu)^2(\lambda+2\mu)^2}$$

$$L_{69} = \frac{3E(\lambda+2\mu)}{4\mu(\lambda+\mu)} (L_{51}-L_{58}) = -\frac{9\lambda^2+24\lambda\mu+32\mu^2}{10(\lambda+\mu)(\lambda+2\mu)} + \frac{2E\lambda W_{20}}{(\lambda+\mu)(\lambda+2\mu)} + \frac{3E\lambda W_{10}}{2\mu(\lambda+\mu)}$$

$$L_{70} = \frac{3E(\lambda+2\mu)}{4\mu(\lambda+\mu)} (L_{52}-L_{59}) = - \frac{27\lambda+34\mu}{10(\lambda+\mu)} - \frac{2EW_{20}}{\lambda+\mu} - \frac{3E(\lambda+2\mu)}{2\mu(\lambda+\mu)} W_{10}$$

$$L_{71} = \frac{3E(\lambda+2\mu)}{4\mu(\lambda+\mu)} (L_{53} - L_{60}) = \frac{3\lambda+\mu}{5(\lambda+\mu)} - \frac{2EW_{20}}{\lambda+\mu} - \frac{3E(\lambda+2\mu)}{2\mu(\lambda+\mu)} W_{10}$$

$$L_{72} = \frac{3E(\lambda+2\mu)}{4\mu(\lambda+\mu)} (L_{54}-L_{61})$$

$$= \frac{2(14\lambda^2+18\lambda\mu+5\mu^2)}{5(\lambda+\mu)(\lambda+2\mu)} - E \frac{4\mu(3\lambda^2+2\lambda\mu+4\mu^2)W_{20}+48\mu^3W_{40}-3\lambda(3\lambda^2+4\lambda\mu+4\mu^2)W_{10}}{2\mu(\lambda+\mu)(\lambda+2\mu)^2}$$

$$L_{73} = \frac{3E(\lambda+2\mu)}{4\mu(\lambda+\mu)} (-L_{33}-L_{62}) = \frac{23\lambda+41\mu}{10(\lambda+\mu)} - \frac{EW_{20}}{\lambda+\mu} + \frac{3E(\lambda+2\mu)}{4\mu(\lambda+\mu)} W_{10}$$

$$L_{74} = \frac{3E(\lambda+2\mu)}{4\mu(\lambda+\mu)} (L_{55}-L_{63})$$

$$= - \frac{39\lambda^2+160\lambda\mu+104\mu^2}{40(\lambda+\mu)^2} + E \frac{8\mu(2\lambda^2+8\lambda\mu+5\mu^2)W_{20}-24\mu^3W_{40}+3\lambda^3W_{10}}{8\mu(\lambda+\mu)^2(\lambda+2\mu)}$$

$$L_{75} = \frac{3E(\lambda+2\mu)}{4\mu(\lambda+\mu)} (L_{56}-L_{64}) = - \frac{\lambda(53\lambda^2+82\lambda\mu+12\mu^2)}{40(\lambda+\mu)^2(\lambda+2\mu)}$$

$$+ E \frac{8\lambda\mu(2\lambda^2+2\lambda\mu-\mu^2)W_{20}+24\mu^3(3\lambda+4\mu)W_{40}-3\lambda^3(3\lambda+4\mu)W_{10}}{8\mu(\lambda+\mu)^2(\lambda+2\mu)^2}$$

$$L_{76} = \frac{\lambda+2\mu}{2(\lambda+\mu)}\left(\frac{3\lambda+2\mu}{\lambda+2\mu} L_{19}+ \frac{\lambda}{\lambda+2\mu} L_{22}-L_{21}\right) = \frac{\lambda E}{2(\lambda+\mu)}\left(\frac{4W_{20}}{\lambda+2\mu} + \frac{3W_{10}}{\mu}\right) - \frac{\lambda+2\mu}{2(\lambda+\mu)}$$

$$L_{77} = \frac{\lambda+2\mu}{4(\lambda+\mu)} (3-2L_{19}) = \frac{5(\lambda+2\mu)}{4(\lambda+\mu)} - \frac{EW_{20}}{\lambda+\mu}$$

$$L_{78} = \frac{\lambda+2\mu}{4(\lambda+\mu)} \left[\frac{2\lambda}{\lambda+2\mu} L_{21} - 2 L_{20} - \left(\frac{3\lambda+2\mu}{\lambda+2\mu}\right)^2\right]$$

$$= - \frac{3\lambda+2\mu}{4(\lambda+\mu)} - \frac{E}{\mu} \frac{4\lambda\mu(\lambda-4\mu)W_{20}+48\mu^3W_{40}-3\lambda^2(\lambda-2\mu)W_{10}}{4(\lambda+\mu)(\lambda+2\mu)^2}$$

$$L_{79} = - \frac{\lambda+2\mu}{2(\lambda+\mu)}(L_{22}+1) = - \frac{\lambda+2\mu}{2(\lambda+\mu)} \left(\frac{3EW_{10}}{2\mu} + 3\right)$$

$$L_{80} = - \frac{\lambda+2\mu}{2(\lambda+\mu)} L_{19} = \frac{\lambda+2\mu}{2(\lambda+\mu)} - \frac{EW_{20}}{\lambda+\mu}$$

$$L_{81} = \frac{\lambda+2\mu}{4(\lambda+\mu)} [2L_{19} - \frac{3(\lambda+2\mu)}{\lambda+\mu} L_{24}+6]$$

$$= \frac{7\lambda^2+12\lambda\mu+8\mu^2}{4(\lambda+\mu)^2} - \frac{E}{4\mu} \frac{4\lambda\mu(\lambda-\mu)W_{20}+24\mu^3W_{40}-3\lambda^3W_{10}}{(\lambda+\mu)^2(\lambda+2\mu)}$$

$$L_{82} = \frac{\lambda+2\mu}{4(\lambda+\mu)} [\frac{2\lambda}{\lambda+2\mu} - \frac{\lambda+2\mu}{2(\lambda+\mu)} L_{20}]$$

$$= \frac{\lambda(7\lambda+6\mu)}{8(\lambda+\mu)^2} - \frac{E}{8\mu} \frac{4\lambda\mu(\lambda-\mu)W_{20}+24\mu^3W_{40}-3\lambda^3W_{10}}{(\lambda+\mu)^2(\lambda+2\mu)}$$

$$L_{83} = \frac{\lambda+2\mu}{4(\lambda+\mu)} [\frac{\lambda+2\mu}{2(\lambda+\mu)} L_{20} - \frac{3\lambda}{\lambda+\mu} L_{24} + \frac{4\lambda}{\lambda+2\mu}]$$

$$= \frac{\lambda(11\lambda^2+16\lambda\mu+12\mu^2)}{8(\lambda+\mu)^2(\lambda+2\mu)} - \frac{E}{\mu} \frac{12\lambda\mu(\lambda^2+\lambda\mu+2\mu^2)W_{20}+(\lambda-2\mu)(24\mu^3W_{40}-3\lambda^3W_{10})}{8(\lambda+\mu)^2(\lambda+2\mu)^2}$$

$$L_{84} = - \frac{\lambda+2\mu}{4(\lambda+\mu)} [2L_{23} + 2L_{25} + \frac{4(\lambda+\mu)}{\lambda+2\mu}]$$

$$= - \frac{11\lambda+14\mu}{6(\lambda+\mu)} - \frac{2EW_{20}}{3(\lambda+\mu)} - \frac{E(\lambda+2\mu)}{2\mu(\lambda+\mu)} W_{10}$$

$$L_{85} = - \frac{\lambda+2\mu}{2(\lambda+\mu)} L_{23} = - \frac{\lambda+3\mu}{6(\lambda+\mu)} - \frac{EW_{20}}{3(\lambda+\mu)} - \frac{E(\lambda+2\mu)}{4\mu(\lambda+\mu)} W_{10}$$

$$L_{86} = - \frac{\lambda+2\mu}{2(\lambda+\mu)} (L_{24}+L_{25}) = - \frac{\lambda^2+13\lambda\mu+10\mu^2}{6(\lambda+\mu)(\lambda+2\mu)}$$

$$- \frac{E}{12\mu} \frac{4\mu(5\lambda^2+8\lambda\mu+8\mu^2)W_{20}+48\mu^3W_{40}+3(-\lambda^3+6\lambda^2\mu+12\lambda\mu^2+8\mu^3)W_{10}}{(\lambda+\mu)(\lambda+2\mu)^2}$$

References

[1] John, F., Estimates for the derivatives of the stresses in a thin shell and interior shell equations, Comm. Pure Appl. Math., Vol. 18, 1965, pp. 235-267.

John, F., Refined interior equations for thin elastic shells, to appear in Comm. Pure Appl. Math.

[2] Green, A. E. and Zerna, W., Theoretical elasticity, Clarendon Press, Oxford, 1954.

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13 ABSTRACT

In this report estimates for the stresses and strains and their derivatives are derived for a thin plate. An isotropic material having a strain energy density function which is an analytic function of the strain invariants is considered, and the estimates are derived from the exact three dimensional theory of elasticity. These estimates are then used to derive approximate interior plate equations with error estimates.

The estimates obtained are of the same type as those derived by F. John [1] for shells, but the work here differs from that in [1] in that consideration is restricted to plates and two strain parameters are used rather than one.

The strain parameters ε_1 and ε_2 are chosen so that when the deformed plate is nearly flat ε_1 is small relative to ε_2 and for relatively large bending ε_2 is small relative to ε_1. Consequently, various simplifications in the plate equations are possible for either small bending or relatively large bending. Also certain vacuities in estimates are avoided by using the two strain parameters.